The Explicit Formula for Fibonacci Sequence

First, let's write out the recursive formula:

$$a_{n+2} = a_{n+1} + a_n$$

where $a_1 = 1, \quad a_2 = 1$

Now, the expression will be modified in 2 different, but similar, ways.

First modification:

$$a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$$

Define a new sequence like this:

$$b_n = a_{n+1} - \alpha a_n$$

Then, the modified version of the Fibonacci Sequence looks like this:

$$b_{n+1} = \beta b_n$$

As you can see, it's telling us that the sequence b_n is a geometric progression, so we can write the explicit form of b_n like this:

$$b_n = b_1 \beta^{n-1} = (a_{1+1} - \alpha a_1)\beta^{n-1} = (a_2 - \alpha a_1)\beta^{n-1} = (1-\alpha)\beta^{n-1}$$

Second modification:

$$a_{n+2} - \beta a_{n+1} = \alpha(a_{n+1} - \beta a_n)$$

Define a new sequence like this:

$$c_n = a_{n+1} - \beta a_n$$

Then, the modified version of the Fibonacci Sequence looks like this:

$$c_{n+1} = \alpha c_n$$

As you can see, it's telling us that the sequence c_n is a geometric progression, so we can write the explicit form of c_n like this:

$$c_n = c_1 \alpha^{n-1} = (a_{1+1} - \beta a_1)\alpha^{n-1} = (a_2 - \beta a_1)\alpha^{n-1} = (1-\beta)\alpha^{n-1}$$

Now, recall that the original version of the Fibonacci Sequence was $a_{n+2} = a_{n+1} + a_n$. Rewrite it as

$$1a_{n+2} - 1a_{n+1} - 1a_n = 0$$

Our **first modified version of the Fibonacci Sequence** was $a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$, and our **second modified version of the Fibonacci Sequence** was $a_{n+2} - \beta a_{n+1} = \alpha(a_{n+1} - \beta a_n)$

Both of them can be rewritten as

$$1a_{n+2} - (\alpha+\beta)a_{n+1} + \alpha\beta a_n = 0$$

The above 2 equations have to be the same, so we can say that

$$\alpha\beta = -1, \quad \alpha+\beta = 1 \Longrightarrow \alpha = 1-\beta \quad \textit{and} \quad \beta = 1-\alpha$$

Substituting it into the 2 modifications yields...

First Modification:

$$b_n = (1-\alpha)\beta^{n-1} = \beta \times \beta^{n-1} = \beta^n$$

Recalling our definition of b_n,

$$a_{n+1} - \alpha a_n = \beta^n$$

Second Modification:

$$c_n = (1-\beta)\alpha^{n-1} = \alpha \times \alpha^{n-1} = \alpha^n$$

(We're almost done) Subtracting those 2 equations gives us

$$\{a_{n+1}-\alpha a_n\}-\{a_{n+1}-\beta a_n\}=\beta^n-\alpha^n \longrightarrow (\beta-\alpha)a_n=\beta^n-\alpha^n$$
$$\therefore a_n=\frac{\beta^n-\alpha^n}{\beta-\alpha}$$

which is the explicit form of the Fibonacci Sequence.

If we can find the values of α and β, then we'll be done. With a bit of thinking, this is quite easy.

Remember that $\alpha\beta=-1, \quad \alpha+\beta=1$ This is telling us that α and β are the roots of $x^2-x-1=0$ since the sum is 1 and the product is -1. Solving it yields

$$\alpha=\frac{1-\sqrt{5}}{2}, \quad \beta=\frac{1+\sqrt{5}}{2}$$

Finally, substitute the values of α and β into the explicit formula.

$$a_n=\frac{\beta^n-\alpha^n}{\beta-\alpha}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n}{\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

#FibonacciNumbers #Notes

Note by **Nick Lee**
6 years, 10 months ago

Comments

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@Calvin Lin I learned this method from my math teacher, but is there a much easier way to derive the explicit formula for the Fibonacci Sequence?

Nick Lee - 6 years, 9 months ago

1 Reply

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찬홍 민 - 6 years, 8 months ago

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