The Bernoulli Numbers: A Brief Primer

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Abstract

In this primer, we explore the diverse properties of a rational sequence known as the Bernoulli numbers. Since the discovery of the numbers in the early eighteenth century, mathematicians have uncovered a vast web of connections between them and core branches of mathematics. We begin with an overview of the historical developments leading to the derivation of the Bernoulli numbers, then use a process similar to that of Jakob Bernoulli to derive the sequence, and finally consider a variety of applications. We hope, above all, to demonstrate how useful and unexpected mathematics can be.

0 Introduction

"Of the various special kinds of numbers used in analysis, there is hardly a species so important and so generally applicable as the Bernoulli numbers."

— David Eugene Smith, *A Source Book in Mathematics*, 1929

The Bernoulli numbers are the terms of a sequence of rational numbers discovered independently by the Swiss mathematician Jakob Bernoulli and Japanese mathematician Seki Takakazu [6]. Both encountered the numbers accidentally in their efforts to calculate the sums of integer powers, $1^m + 2^m + \cdots + n^m$. Since this discovery, the Bernoulli numbers have appeared in many important results, including the series expansions of trigonometric and hyperbolic trigonometric functions, the Euler-Maclaurin Summation Formula, the evaluation of the Riemann zeta function, and Fermat's Last Theorem.

This primer is intended to spark the reader's interest. To that end, we briefly discuss the history of the mathematics that led to the sequence's discovery and then touch on a wide variety of applications of the Bernoulli numbers. We hope to show this sequence is not only surprising, but also a useful tool in a variety of core problems in mathematics.

The primer is laid out as follows. Sections 1 and 2 outline the historical developments leading up to the discovery of the Bernoulli sequence. Section 3 defines the Bernoulli numbers as we see them today, as coefficients of a generating function, and in section 4, we make some preliminary observations about the sequence. The body of the primer, laid out in sections 5 to 14, explores applications of the Bernoulli numbers to various fields of mathematics. We end in section 14 with an exciting application to Fermat's Last Theorem, before our concluding remarks in section 15. One appendix discusses notation and definition issues and another includes a list of Bernoulli numbers for reference.

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1 Uncovering the Bernoulli Numbers: A History

The Bernoulli numbers were discovered in the process of solving an ancient problem. Both Jakob Bernoulli and Seki Takakazu stumbled across the sequence while trying to find a general formula for the "sums of integer powers," defined as

$$S_m(n) = 1^m + 2^m + 3^m + \cdots + n^m$$

for positive integers n and m. In this section, we examine the the work of other prior mathematicians on the problem. The story is both fascinating and useful to us, because it (1) demonstrates how elusive the sequence was, even to some of the great mathematical minds of each era, (2) how mathematics has been (not) communicated throughout history, and (3) how we, as mathematicians and historians, choose to memorialize some important figures and not others.

Since antiquity, mathematicians struggled to sum the integer powers [1]. In some cases, the mathematicians were driven by curiosity. Others needed formulas to solve specific problems in engineering and physics. Many mathematical minds made progress on the problem, but a general formula proved elusive.

Archimedes of Syracuse (287-212 BC), the greatest mathematician of antiquity and perhaps of all time [6], is one of the first people on record to have considered solving for the sums of integer powers. In the fashion of many great mathematicians, he may have discovered a formula for the sums of squares, but didn't formally state it. Rather, he simply used it as a step in another proof.

Aryabhata (b. 476), a major early physicist and astronomer in India, discovered a formula for the sums of cubes. Abu Bakr Al-Karaji of Baghdad (d. 1019), an engineer and mathematician, wrote out the sums of cubes up to 10. It is likely that he derived a formula as well. In the next decade, the Iraqi mathematician Abu Ali al-Hassan ibn al Haytham (965-1039)–known as "Alhazen" in Europe–wrote his magnum opus, the seven-volume *Optics*, which included a result that required knowledge that $S_4(n) = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$. This knowledge was not passed on, because he did not specifically state the result.

European mathematicians rediscovered methods to calculate the sums of fourth powers much later–in the sixteenth and seventeenth centuries–but took the results significantly further. Thomas Harriot (1560-1621), a mathematician and scientist under the patronage of Sir Walter Raleigh, used "difference tables" to calculate the sums of fourth powers on a voyage to the colony of Virginia in 1585. He also introduced new symbolic notation, a development that put him well ahead of many of his contemporaries, who still wrote out all of their mathematical calculations in sentences. Unfortunately, Harriot never published the 5000 pages of mathematical notes he wrote and only passed on some of his knowledge through letters.

Pierre de Fermat (1601-1665), a French lawyer, discovered his own formula for the sums of fourth powers, which he used to compute definite integrals of the form cx^k. Like Harriot, he never published his work, but instead corresponded regularly with several other amateur and professional mathematicians. Historians believe that his technique of using power series to determine area inspired Newton as he developed a framework for calculus in the period from 1665 to 1670.

Blaise Pascal (1623-1662) built upon the advances of his predecessors. He introduced his now-famous Arithmetical Triangle to the problem to use previous sums of powers to calculate the next. This system theoretically allowed the computation of every sum of powers, but in practice it quickly became too convoluted to accurately use.

What both Fermat and Pascal missed, say historical scholars, was the work of Johann Faulhaber (1580-1635). Born into a family of basket-weavers in Ulm, Germany, Faulhaber displayed a great talent for computation. He started a mathematics school in his hometown, which became well-known to a number of mathematicians (although clearly not all), including René Decartes, who sought him out as a tutor. In 1610, Faulhaber made significant strides on the problem of the sums of integer powers by calculating explicit formulas for sums up to the tenth power. In *Academia Algebrae*, his 1631 masterwork, he gave formulas for powers up to twenty-three.

Faulhaber, as a matter of historical interest, was also quite eccentric. He believed in what he called "figured numbers" from the Bible, and used them to try to predict future events. In fact, he was jailed for predicting the end of the world in 1605. He claimed later he also could convert lead into gold. Despite these eccentricities, or perhaps because of them, the serious mathematical work of Faulhaber was relatively unknown during his time, and remains so today. Only one surviving copy of *Academia Algebrae* is known to exist today.

Despite the lineage of European mathematicians who worked on the sums of powers for centuries, it was actually a Japanese mathematician, Seki Takakazu (1642-1708), who first discovered the "Bernoulli numbers." Seki was born in Fujioka Gumma, Japan to a samurai warrior family. From an early age, Seki demonstrated prodigious mathematical talent, and in his later years he is credited with transforming the study of mathematics in Japan. In 1683, Seki became the first mathematician to study determinants (before Leibniz), and used them to solve more general equations than Leibniz did ten years later. Seki had a method analogous to Newton polynomial interpolation and solved cubic polynomials using a method not yet discovered in Europe. Furthermore, using a technique called *Ruisai Shosa-ho*, he discovered the sequence of the Bernoulli numbers and their role in computing the sums of powers.

Halfway around the world, Jakob Bernoulli (1655-1705) was born in Basel, Switzerland to a family of merchants. If the name Bernoulli sounds familiar, it should. Within two generations in the seventeenth century, the Bernoulli family produced a dozen prominent mathematicians and scientists. For example, the famous "Bernoulli Principle" in physics, which describes how fast-moving air over a surface generates lift, was named for Jakob Bernoulli's nephew, Daniel, the son of Jakob's brother (and rival) Johann. Jakob Bernoulli discovered the number $e = 2.718\ldots$, developed the beginnings of a theory of series and proved the law of large numbers in probability theory, but contributed most significantly to mathematics with his work *Ars Conjectandi*. In this work, he laid out his solutions to the first ten sums of powers, and the sequence of numbers he uncovered during his calculations.

These two mathematicians were situated in very different cultural and mathematical worlds, but managed to uncover this important sequence of numbers at nearly the same time. We appreciate contemporary mathematicians who refer to the numbers as the Seki-Bernoulli numbers, because it follows the convention to give both discoverers credit in the case of independent discovery (e.g. the Euler-Maclaurin Summation Formula, the Calusen-von Staudt Theorem). In this primer, we choose to call the sequence the "Bernoulli numbers" to increase readability (although this may change). We also acknowledge that the body of work developed using the Bernoulli numbers was inspired largely by the work of Bernoulli rather than Seki. However, this comes after significant consideration, and I do not believe this is the best or only conclusion to reach.

I find it important to note the history of mathematics is not equitable. History is not what happened, but merely what has been recorded, and most of what has been recorded in English has a distinctly Western bent. This is particularly true in the field of mathematical history. Records emphasize Greek, German, English, French, Russian, Italian, and other European contributions while neglecting major work from other parts of the world. As Leigh Wood states in *Mathematics Across Cultures: The History of Non-Western Mathematics*, "mathematics itself is not one culture with one discourse" [19]. That is why I find it important, for example, to recognize the work of early Indian, Egyptian, and Iraqi mathematicians to the problem of the sums of powers. The West has dominated mathematical thought for the past few centuries, but before that, it was Asian, Middle Eastern, and even pre-Colombian American cultures that drove mathematical discovery for millennia. Even during the height of European mathematics, significant contributions were made by non-Europeans who have not been properly recognized. Such is the case of Seki Takakazu.

After Seki Takakazu and Bernoulli independently discovered the sequence of numbers in the early eighteenth century, mathematicians began to find connections between the sequence and many mathematical fields. Among the contributors to this body of research are a number of familiar names in (European) mathematical history, among them: Abraham de Moivre (1667-1754), Colin Maclaurin (1698-1745), Karl Georg Christian von Staudt (1798-1867), Ernst Edward Kummer (1810-1893), Adrien-Marie Legendre (1752-1833), Peter Dirichlet (1805-1859), and Georgii Voronoi (1868-1908). In this primer, we will dedicate significant time to the results of Léonard Euler (1707-1783), who was one of the first to study the sequence in depth after Bernoulli's publication. I will note that no work building off of Seki's discovery was found in my research. Perhaps his discovery was not shared widely, or the appropriate historical records are simply not accessible in English.

2 Following in Bernoulli's Footsteps: Sums of Powers

Seki Takakazu's method for finding the Bernoulli numbers is not easily converted to Western notation, so let us derive the sequence by Jakob Bernoulli's method. Bernoulli's process was not unlike those of some of his predecessors, but he made several keen observations that led to a final solution. In this section, we approximately retrace his steps.

In *Ars Conjectandi*, Bernoulli calculated the formulas for $S_m(n)$ up to ten using the methods of Fermat [1]. Here, we have listed the first six sums up to the integer $n-1$, because this form allows us more clearly to see useful patterns:

$$\begin{aligned}
S_1(n-1) &= \frac{1}{2}n^2 && - \frac{1}{2}n \\
S_2(n-1) &= \frac{1}{3}n^3 && - \frac{1}{2}n^2 && + \frac{1}{6}n \\
S_3(n-1) &= \frac{1}{4}n^4 && - \frac{1}{2}n^3 && + \frac{1}{4}n^2 \\
S_4(n-1) &= \frac{1}{5}n^5 && - \frac{1}{2}n^4 && + \frac{1}{3}n^3 && - \frac{1}{30}n \\
S_5(n-1) &= \frac{1}{6}n^6 && - \frac{1}{2}n^5 && + \frac{5}{12}n^4 && - \frac{1}{12}n^2 \\
S_6(n-1) &= \frac{1}{7}n^7 && - \frac{1}{2}n^6 && + \frac{1}{2}n^5 && - \frac{1}{6}n^3 && + \frac{1}{42}n
\end{aligned}$$

We can observe, as he did, that the leading term of the formula for each $S_m(n-1)$ is $\frac{1}{m+1}n^{m+1}$. Bernoulli factored out the fraction $\frac{1}{m+1}$ from each polynomial and obtained the following chart:

$$\begin{aligned}
S_1(n-1) &= \frac{1}{2}\Big[n^2 \quad - n\Big] \\
S_2(n-1) &= \frac{1}{3}\Big[n^3 \quad - \frac{3}{2}n^2 \quad + \frac{1}{2}n\Big] \\
S_3(n-1) &= \frac{1}{4}\Big[n^4 \quad - 2n^3 \quad + n^2\Big] \\
S_4(n-1) &= \frac{1}{5}\Big[n^5 \quad - \frac{5}{2}n^4 \quad + \frac{5}{3}n^3 \quad - \frac{1}{6}n\Big] \\
S_5(n-1) &= \frac{1}{6}\Big[n^6 \quad - 3n^5 \quad + \frac{5}{2}n^4 \quad - \frac{1}{2}n^2\Big] \\
S_6(n-1) &= \frac{1}{7}\Big[n^7 \quad - \frac{7}{2}n^6 \quad + \frac{7}{2}n^5 \quad - \frac{7}{6}n^3 \quad + \frac{1}{6}n\Big]
\end{aligned}$$

The next observation he made was more subtle. Consider the second column of terms. Each is a multiple of $\frac{1}{2}$: $-1 = 2 \cdot -\frac{1}{2}, -\frac{3}{2} = 3 \cdot -\frac{1}{2}, -2 = 4 \cdot -\frac{1}{2}$, and so on. The third column contains multiples of $\frac{1}{6}$, and the fourth, multiples of $-\frac{1}{30}$. If we factor these terms out of each formula, get obtain,

$$\begin{aligned}
S_1(n-1) &= \frac{1}{2}\Big[n^2 \quad - 2\left(\frac{1}{2}\right)n\Big] \\
S_2(n-1) &= \frac{1}{3}\Big[n^3 \quad - 3\left(\frac{1}{2}\right)n^2 \quad + 3\left(\frac{1}{6}\right)\Big] \\
S_3(n-1) &= \frac{1}{4}\Big[n^4 \quad - 4\left(\frac{1}{2}\right)n^3 \quad + 6\left(\frac{1}{6}\right)\Big] \\
S_4(n-1) &= \frac{1}{5}\Big[n^5 \quad - 5\left(\frac{1}{2}\right)n^4 \quad + 10\left(\frac{1}{6}\right)n^3 \quad - 5\left(\frac{1}{30}\right)n\Big] \\
S_5(n-1) &= \frac{1}{6}\Big[n^6 \quad - 6\left(\frac{1}{2}\right)n^5 \quad + 15\left(\frac{1}{6}\right)n^4 \quad - 15\left(\frac{1}{30}\right)n^2\Big] \\
S_6(n-1) &= \frac{1}{7}\Big[n^7 \quad - 7\left(\frac{1}{2}\right)n^6 \quad + 21\left(\frac{1}{6}\right)n^5 \quad - 35\left(\frac{1}{30}\right)n^3 \quad + 7\left(\frac{1}{42}\right)n\Big]
\end{aligned}$$

Our next step arises unexpectedly. If we look at the integer coefficients that remain for each term, we may notice what Bernoulli did: that the integers are all binomial coefficients corresponding to the row of m. It looks almost like Pascal's Triangle, with a few gaps that we can replace by terms multiplied by zero. If we write the formulas out in this way, we get

$$S_1(n-1) = \frac{1}{2}\left[\binom{2}{0}n^2 \quad - \binom{2}{1}\frac{1}{2}n\right]$$
$$S_2(n-1) = \frac{1}{3}\left[\binom{3}{0}n^3 \quad - \binom{3}{1}\frac{1}{2}n^2 \quad + \binom{3}{2}\frac{1}{6}n\right]$$
$$S_3(n-1) = \frac{1}{4}\left[\binom{4}{0}n^4 \quad - \binom{4}{1}\frac{1}{2}n^3 \quad + \binom{4}{2}\frac{1}{6}n^2 \quad + \binom{4}{3}0n\right]$$
$$S_4(n-1) = \frac{1}{5}\left[\binom{5}{0}n^5 \quad - \binom{5}{1}\frac{1}{2}n^4 \quad + \binom{5}{2}\frac{1}{6}n^3 \quad + \binom{5}{3}0n^2 \quad - \binom{5}{4}\frac{1}{30}n\right]$$
$$S_5(n-1) = \frac{1}{6}\left[\binom{6}{0}n^6 \quad - \binom{6}{1}\frac{1}{2}n^5 \quad + \binom{6}{2}\frac{1}{6}n^4 \quad + \binom{6}{3}0n^3 \quad - \binom{6}{4}\frac{1}{30}n^2 \quad + \binom{6}{5}0n\right]$$
$$S_6(n-1) = \frac{1}{7}\left[\binom{7}{0}n^7 \quad - \binom{7}{1}\frac{1}{2}n^6 \quad + \binom{7}{2}\frac{1}{6}n^5 \quad + \binom{7}{3}0n^4 \quad - \binom{7}{4}\frac{1}{30}n^3 \quad + \binom{7}{5}0n^2 \quad + \binom{7}{6}\frac{1}{42}n\right]$$

At this point, we suddenly see a pattern that connects all of the formulas. In fact, we can write out a compact equation, one that we prove in section 9:

$$S_m(n-1) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m-k+1} \tag{1}$$

where B_k is the mysterious sequence of numbers that we factored out in the second step,

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 - \frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42},\ B_7 = 0, \ldots$$

These are the Bernoulli numbers.

Bernoulli realized that this sequence was valuable to the problem he solved, but did not do further work with the numbers. After Bernoulli's publication, the next mathematician to work with the sequence in depth was Leonard Euler (1707-1783). He used the Bernoulli numbers to derive a solution to the even values of the zeta function and develop a general summation formula (the Euler-Maclaurin Summation Formula) in the 1730s. However, he did not introduce his now-standard definition for the sequence until two decades later, perhaps due to the "lack of any obvious pattern among the Bernoulli numbers" [6]. Even to Euler, these numbers were mysterious and difficult to pin down.

3 The Bernoulli Generating Function

In 1755, Euler posed the following definition for the Bernoulli numbers, which remains the most common modern definition for the sequence.

Definition 3.1. The *Bernoulli numbers* are the coefficients of the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty}\frac{B_k x^k}{k!}.$$

Euler's formal definition for the Bernoulli numbers is based on the concept of a generating function, which is a method of encoding a sequence. To review: we say that f is an "ordinary" generating function for the sequence $\{a_n\}_{n=0}^{\infty}$ if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i.$$

Similarly, f is an "exponential" generating function for $\{b_n\}_{n=0}^{\infty}$ if

$$f(x) = b_0 + b_1 \frac{x}{1!} + b_2 \frac{x^2}{2!} + \cdots = \sum_{i=0}^{\infty} b_i \frac{x^i}{i!}.$$

So, a function that is the ordinary generating function of $\{a_n\}$ is the exponential generating function of $\{b_n\} = n! \cdot \{a_n\} = \{n! \cdot a_n\}$. We use whichever function is more appropriate for a particular problem. In this case, the exponential generating function is appropriate because of the the exponential term e^x in the denominator.

We can work out the first few terms of the sequence by evaluating the Taylor series expansion of $\frac{x}{e^x-1}$. We can calculate the first couple derivatives and their limits as x approaches 0,

$$\begin{aligned} f(x) &= \frac{x}{e^x - 1} & &\text{with } \lim_{x \to 0} f(x) = 1 \\ f'(x) &= \frac{e^x x - e^x + 1}{(e^x - 1)^2} & &\text{with } \lim_{x \to 0} f'(x) = -\frac{1}{2} \\ f''(x) &= \frac{e^{2x} x - e^x x - 2e^{2x} + 2e^x}{(e^x - 1)^3} & &\text{with } \lim_{x \to 0} f''(x) = \frac{1}{6} \\ &\cdots & &\cdots \end{aligned}$$

which gives us the Taylor series centered at 0 (aka Macluarin series):

$$\begin{aligned} \frac{x}{e^x - 1} &= \sum_{n=0}^{\infty} \frac{f^{(n)} x^n}{n!} \\ &= 1 + \left(-\frac{1}{2}\right)x + \left(\frac{1}{6}\right)\frac{x^2}{2!} + \left(-\frac{1}{30}\right)\frac{x^4}{4} + \left(\frac{1}{42}\right)\frac{x^6}{6!} + \cdots \\ &= 1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^4}{720} + \frac{x^6}{30240} + \cdots \end{aligned}$$

While visualizing the function $\frac{x}{e^x-1}$ does not give us direct insight into the Bernoulli numbers, it is helpful and interesting to know what we are dealing with. Figure 1 displays a graph of the function, which is defined for all $x \neq 0$ and ranges over the positive reals.

Note that while the generating function is defined nearly everywhere, we must consider the radius of convergence of its Taylor series. Any point at which the derivative of a function f does not exist is called a *singularity* or *singular point* of f. The radius of convergence of a power series is the distance from the origin to the nearest singularity of the function that the series represents.

In our case, f' is undefined whenever $e^x = 1$, except at the point $x = 0$. Thus, from Euler's formula we see that the nearest singularities to the origin are at $x = \pm 2\pi i$. Therefore, the radius of convergence is $x < |2\pi|$, the portion of the function shown in Figure 1.

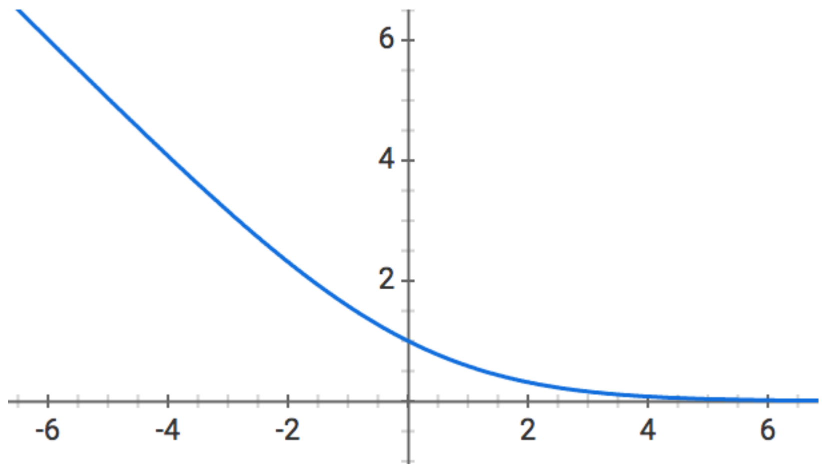

Figure 1: The function $f(x) = \frac{x}{e^x - 1}$

4 Preliminary Observations

Let us step back from the exponential generating function to take a closer look at the Bernoulli numbers themselves. When you look at the first terms of the Bernoulli numbers, what do you notice?

$$
\begin{aligned}
B_0 &= 1 & B_{11} &= 0 \\
B_1 &= -1/2 & B_{12} &= -691/2730 \\
B_2 &= 1/6 & B_{13} &= 0 \\
B_3 &= 0 & B_{14} &= 7/6 \\
B_4 &= -1/30 & B_{15} &= 0 \\
B_5 &= 0 & B_{16} &= -3617/510 \\
B_6 &= 1/42 & B_{17} &= 0 \\
B_7 &= 0 & B_{18} &= 43867/798 \\
B_8 &= -1/30 & &\cdots \\
B_9 &= 0 & B_{49} &= 0 \\
B_{10} &= 5/66 & B_{50} &= 495057205241079648212477525/66
\end{aligned}
$$

Likely, a few of the patterns you see are among the following:

1. B_n is rational.
2. $B_{2n+1} = 0$ for $n \geq 1$.
3. B_{2n} alternates sign: $B_{4n} < 0$ and $B_{4n+2} > 0$ for $n \geq 1$.
4. The magnitude of B_{2n} grows very quickly.

Each of these observations is true for all of the Bernoulli numbers. The first two observations we prove now. The other two patterns will appear as we delve deeper into the properties of the sequence.

4.1 The Bernoulli Numbers Are Rational

One of the key properties of the Bernoulli numbers is that they are rational. To prove this fact, we will derive the following recurrence relation for the Bernoulli numbers. If the proposition below is true, we note that the fact that B_k is rational follows immediately.

Proposition 4.1. The Bernoulli numbers satisfy the relation

$$B_0 = 1 \text{ and } \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \text{ for } n > 1.$$

Proof. This is our process: we multiply by $e^x - 1$ on both sides, express $e^x - 1$ as a Taylor series, take the Cauchy product of this series with $\sum_{i=0}^{\infty} \frac{B_i x^i}{i!}$, and then equate powers of x. First,

$$\begin{aligned}
\frac{x}{e^x - 1} &= \sum_{i=0}^{\infty} \frac{B_i x^i}{i!} \\
x &= (e^x - 1) \sum_{i=0}^{\infty} \frac{B_i x^i}{i!} \\
&= (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) \sum_{i=0}^{\infty} \frac{B_i x^i}{i!} \\
&= \sum_{j=1}^{\infty} \frac{x^j}{j!} \sum_{i=0}^{\infty} \frac{B_i x^i}{i!} \\
&= \sum_{j=0}^{\infty} \frac{x^{j+1}}{(j+1)!} \sum_{i=0}^{\infty} \frac{B_i x^i}{i!}.
\end{aligned}$$

Recall the Cauchy product of two infinite series:

$$\left(\sum_{k=0}^{\infty} a_k \right) \left(\sum_{m=0}^{\infty} b_m \right) = \left(\sum_{n=0}^{\infty} c_n \right)$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$. If we take the Cauchy product in this case, we obtain

$$\begin{aligned}
x &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{n+1-k}}{(n+1-k)!} \cdot \frac{B_k x^k}{k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{B_k x^{n+1}}{(n+1-k)! k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(n+1)! B_k}{(n+1-k)! k!} \frac{x^{n+1}}{(n+1)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+1}{k} B_k \frac{x^{n+1}}{(n+1)!}.
\end{aligned}$$

Finally, substituting $n - 1$ for n yields the equation

$$x = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{n}{k} B_k \frac{x^n}{n!}.$$

On the left hand side we have only x. We know the coefficient of x on the right hand side is 0, and the coefficient of every other power of x is 0. Thus, the desired relation results:

$$B_0 = 1 \text{ and } \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \text{ for } n > 1.$$

□

This is a valuable recurrence relation. Not only does it prove that the sequence is rational, but it also leads to an intuitive understanding of the structure of the Bernoulli numbers. A mathematics student could be forgiven for asking why the terms of the generating function $\frac{x}{e^x-1}$ are so important. This reformulation captures the fundamental relation of the Bernoulli numbers to one another, which we can demonstrate by writing the first few terms of the recurrence:

$$\begin{aligned}
1 &= B_0 \\
0 &= B_0 + 2B_1 \\
0 &= B_0 + 3B_1 + 3B_2 \\
0 &= B_0 + 4B_1 + 6B_2 + 4B_3 \\
0 &= B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4.
\end{aligned}$$

Perhaps the most memorable way of remembering this is relationship is through the pseudo-equation $(B+1)^n = B^n$, where the left hand side is expanded $(B^n + \binom{n}{n-1}B^{n-1} + \cdots + \binom{n}{1}B^1 + 1 = B^n)$ and then all exponents are converted into subscripts $(B_n + \binom{n}{n-1}B_{n-1} + \cdots + \binom{n}{1}B_1 + 1 = B_n)$. This simple mnemonic proves useful for remembering the Bernoulli numbers, and understanding their close association with the Binomial Theorem and Pascal's Triangle. We will explore this relationship further in our section on matrices.

4.2 The Odd Bernoulli Numbers (Except B_1) Are Zero

All odd Bernoulli numbers aside from $B_1 = -\frac{1}{2}$ are zero. The case of B_1 is interesting, and we consider it specifically in the appendix. But for now, we present a proof for the rest of the odd terms.

Proposition 4.2. $B_{2n+1} = 0$ for all $n \geq 1$.

Proof. Consider the Bernoulli generating function $\frac{x}{e^x-1} = B_0 + B_1 x + \frac{B_2 x^2}{2!} + \frac{B_3 x^3}{3!} + \cdots$ minus the term

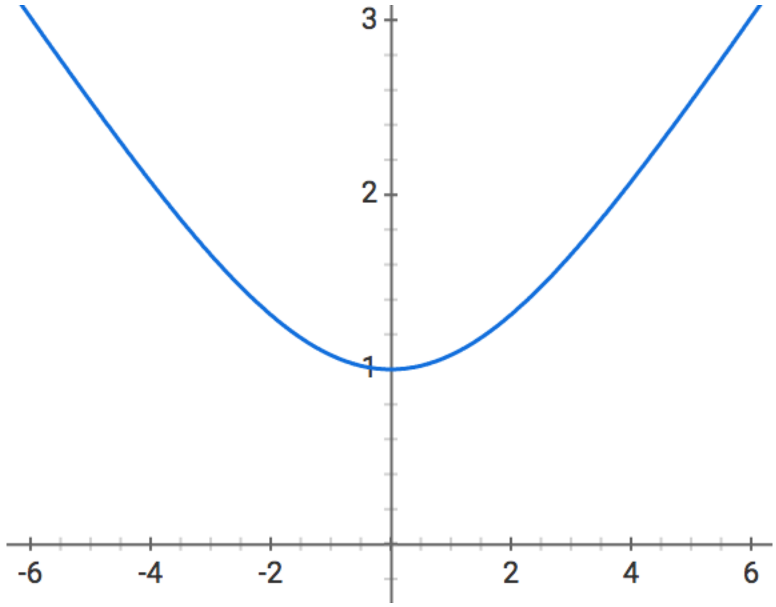

Figure 2: The function $g(x) = \frac{x}{e^x-1} - B_1 x$ is even

$B_1 x$. Then we have

$$\begin{aligned} g(x) &= \frac{x}{e^x - 1} - B_1 x \\ &= \frac{x}{e^x - 1} + \frac{x}{2} \\ &= \frac{2x + x(e^x - 1)}{2(e^x - 1)} \\ &= \frac{x(e^x + 1)}{2(e^x - 1)} \\ &= \frac{x(e^x + 1)}{2(e^x - 1)} \left(\frac{e^{-x/2}}{e^{-x/2}} \right) \\ &= \frac{x(e^{x/2} + e^{-x/2})}{2(e^{x/2} - e^{-x/2})}. \end{aligned}$$

If we plug $-x$ into the right hand side, we obtain

$$\begin{aligned} g(-x) &= \frac{-x(e^{-x/2} + e^{x/2})}{2(e^{-x/2} - e^{x/2})} \\ &= \frac{-x(e^{x/2} + e^{-x/2})}{-2(e^{x/2} - e^{-x/2})} \\ &= g(x). \end{aligned}$$

Therefore, g is even (see Figure 2). Thus, the power series of $\frac{x}{e^x-1} - B_1 x$ has no nonzero odd-power terms, and $B_{2n+1} = 0$ for all $n \geq 1$. □

5 Bernoulli Numbers and Cotangent

We know from the previous section that

$$g(x) = \frac{x}{e^x - 1} - B_1 x = \frac{x(e^{x/2} + e^{-x/2})}{2(e^{x/2} - e^{-x/2})}. \tag{2}$$

Let us explore this equation further. We notice that the right hand side of this equation looks like a hyperbolic trigonometric curve–a trigonometric function in the hyperbolic plane. We know that the hyperbolic sine and cosine curves are expressed by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Thus,

$$\frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = \frac{\frac{e^{x/2}+e^{-x/2}}{2}}{\frac{e^{x/2}-e^{-x/2}}{2}} = \frac{\cosh \frac{x}{2}}{\sinh \frac{x}{2}} = \coth \frac{x}{2}$$

and in equation 2, we notice that

$$\begin{aligned} g(x) &= \frac{x(e^{x/2} + e^{-x/2})}{2(e^{x/2} - e^{-x/2})} \\ &= \frac{x}{2} \coth \frac{x}{2}. \end{aligned}$$

Since the Taylor expansion of the left-hand side has no nonzero odd terms, we may write,

$$\frac{x}{2} \coth \frac{x}{2} = \sum_{n=0}^{\infty} \frac{B_{2n} x^{2n}}{(2n)!}$$

from which we can derive an expression for hyperbolic cotangent:

$$\begin{aligned} \coth x &= \sum_{n=0}^{\infty} \frac{B_{2n}(2x)^{2n}}{x(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{2B_{2n}(2x)^{2n-1}}{(2n)!}. \end{aligned}$$

If we substitute xi for x in this equation, we find an expression in terms of cotangent, for $|x| \leq \pi$

$$\cot x = \sum_{n=0}^{\infty} (-1)^n \frac{2B_{2n}(2x)^{2n-1}}{(2n)!}.$$

Furthermore, $\tan x, \tanh x, \ln \sin x, \frac{x}{\sin x}$ and other trigonometric functions can be expressed in terms of the Bernoulli numbers. These expressions will be useful as we look closer at the Riemann zeta function in the next section.

6 The Riemann Zeta Function

One of the most powerful applications of the Bernoulli numbers the evaluation of the Riemann zeta function.

Definition 6.1. Let k be a real, $|k| \geq 1$. Then the *Riemann zeta function over the real numbers*, $\zeta(k)$, is defined as

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

This function is important for many reasons, but we will highlight one result proven by Euler related to the prime numbers.

Theorem 6.1. *For $k > 1$,*

$$\zeta(k) = \prod_p \left(\frac{1}{1 - p^{-k}} \right)$$

over all primes p.

Proof. The proof of this theorem relies on the uniqueness of factorization guaranteed by the Fundamental Theorem of Arithmetic. There are two approaches–one which involves a sieve and another which follows from geometric series. We will take the latter approach.

For $0 < x < 1$, we have

$$\frac{1}{1-x} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots$$

If, for each prime p and $k > 1$, we say $x = \frac{1}{p^k}$:

$$\frac{1}{1 - \frac{1}{p^k}} = 1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \frac{1}{p^{3k}} + \frac{1}{p^{4k}} + \cdots$$

If we take the product of each of these generating functions on the left hand side, we get

$$\left(\frac{1}{1 - \frac{1}{2^k}} \right) \left(\frac{1}{1 - \frac{1}{3^k}} \right) \left(\frac{1}{1 - \frac{1}{5^k}} \right) \cdots = (1 + \frac{1}{2^k} + \frac{1}{2^{2k}} + \cdots)(1 + \frac{1}{3^k} + \frac{1}{3^{2k}} + \cdots)(1 + \frac{1}{5^k} + \frac{1}{5^{2k}} + \cdots) + \cdots$$

We now employ the FTA. Every term of the expansion on the right hand side will be of the form

$$\frac{1}{p_1^{m_1 k} p_2^{m_2 k} \cdots p_n^{m_n k}}$$

where $m_1, \ldots, m_n$ are positive integers. By the FTA, each positive integer has a unique factorization into the powers of primes. Therefore the expansion becomes,

$$\left(\frac{1}{1 - \frac{1}{2^k}} \right) \left(\frac{1}{1 - \frac{1}{3^k}} \right) \left(\frac{1}{1 - \frac{1}{5^k}} \right) \cdots = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + \frac{1}{6^k} + \cdots$$
$$\prod_p \left(\frac{1}{1 - p^{-k}} \right) = \zeta(k).$$

□

The result is a beautiful and unexpected connection between the zeta function and the primes, and is related to the famous prime number theorem, which describes the distribution of prime numbers in the positive integers.

The Bernoulli numbers help us to calculate the even values of this function. The two key parts of the proof are an infinite polynomial for $\sin x$ and the formula for $\cot x$ that we derived above.

Theorem 6.2. *For any integer $k > 0$,*

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}.$$

It is remarkable to see the Bernoulli numbers appear in this formula, as well as to see that even values of the zeta function are rational numbers multiplied by powers of π. We will prove this theorem by equating two different expressions for cotangent. But first, we will give ourselves an intuition regarding this problem by considering the case $\zeta(2)$, which Euler solved equating two expressions for *sine*. We start with the following lemma:

Lemma 6.1. The function $\sin x$ can be written as the infinite polynomial

$$\sin x = \lim_{n\to\infty} x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots\left(1-\frac{x}{n\pi}\right)\left(1+\frac{x}{n\pi}\right).$$

In this paper, we take this formula at face value and avoid the heavy machinery of complex analysis (refer to [13] for more information). Instead, we reference Euler, who supposed (correctly) that since the roots of $\sin x$ are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$, that the function could be written as an infinite polynomial with the above form. With this polynomial expression for $\sin x$ in hand, we follow in Euler's footsteps to calculate the value of $\zeta(2)$.

$$\begin{aligned}
\sin x &= \lim_{n\to\infty} x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots\left(1-\frac{x}{n\pi}\right)\left(1+\frac{x}{n\pi}\right)\\
&= \lim_{n\to\infty} x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots\left(1-\frac{x}{n^2\pi^2}\right)\\
\frac{\sin x}{x} &= \lim_{n\to\infty} \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots\left(1-\frac{x}{n^2\pi^2}\right)\\
&= \prod_{n=1}^{\infty}\left(1-\left(\frac{x}{n\pi}\right)^2\right).
\end{aligned}$$

What if we try to isolate the x^2 term? In order to obtain a x^2 term, we simply take one of the $-\frac{x^2}{k\pi^2}$ multiplied by all ones in the expansion of $\frac{\sin x}{x}$. So, we see that

$$\begin{aligned}
\frac{\sin x}{x} &= 1 - \frac{x^2}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots\right) + O(x^4)\\
&= 1 - \frac{x^2}{\pi^2}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right) + O(x^4)\\
&= 1 - \frac{x^2}{\pi^2}\zeta(2) + O(x^4)
\end{aligned}$$

where $O(x^4)$ represents terms of x with degree greater than or equal to 4. Now we consider another expression of $\sin x$. In addition to the formula we just derived, we know the Taylor series of $\sin x$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

from which we can calculate

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Now, if we compare the coefficients of x^2 in each equation, we see that

$$\begin{aligned} \frac{1}{\pi^2}\zeta(2) &= \frac{1}{3!} \\ \zeta(2) &= \frac{\pi^2}{6}. \end{aligned}$$

This is a fascinating result. But how do we generalize our function in order to find larger values of the zeta function, such as $\zeta(10)$? The secret is in the cotangent formula in terms of the Bernoulli numbers that we derived in the previous section, which we will compare to a cotangent formula in terms of the zeta function. Before we can do that, we must find such a formula.

Consider the infinite polynomial expression for $\frac{\sin x}{x}$. We want to write $\cot x$ in terms of $\sin x$ in order to make a substitution. Recall,

$$\frac{d}{dx} \ln \sin x = \frac{\cos x}{\sin x} = \cot x.$$

If we take the natural logarithm of $\frac{\sin x}{x}$, we see,

$$\begin{aligned} \ln \frac{\sin x}{x} &= \ln \left[\prod_{n=1}^{\infty} \left(1 - \left(\frac{x}{n\pi} \right)^2 \right) \right] \\ \ln \sin x - \ln x &= \ln \left[\prod_{n=1}^{\infty} \left(1 - \left(\frac{x}{n\pi} \right)^2 \right) \right] \\ \ln \sin x &= \ln x + \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{(n\pi)^2} \right). \end{aligned}$$

We want to take the derivative of both sides. Since

$$\frac{d}{dx} \ln \left(1 - \frac{x^2}{(n\pi)^2} \right) = \frac{d}{dx} \ln \left(\frac{(n\pi)^2 - x^2}{(n\pi)^2} \right) = \left(\frac{-2x}{(n\pi)^2 - x^2} \right) = \frac{2x}{x^2 - (n\pi)^2}$$

then when we take the derivative we obtain an expression for cotangent,

$$\begin{aligned} \cot x &= \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{2x}{x^2 - (n\pi)^2} \right) \\ &= \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x + n\pi} + \frac{1}{x - n\pi} \right) \\ &= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n} \left(\frac{1}{1 + \frac{x}{n\pi}} + \frac{1}{1 - \frac{x}{n\pi}} \right). \end{aligned}$$

We are almost ready, but some additional manipulation is required to tease out a term of $\zeta(2n)$. Next, we expand out into the difference of two power series. Since $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ and $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$, we have

$$\begin{aligned}
\cot x &= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\left(\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{n\pi}\right)^k - \sum_{k=0}^{\infty}\left(\frac{x}{n\pi}\right)^k\right) \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\left(\left[\sum_{k=0}^{\infty}\left(\frac{x}{n\pi}\right)^{2k} - \left(\frac{x}{n\pi}\right)^{2k+1}\right] - \sum_{k=0}^{\infty}\left[\left(\frac{x}{n\pi}\right)^{2k} - \left(\frac{x}{n\pi}\right)^{2k+1}\right]\right) \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\left(2\sum_{k=0}^{\infty}\left(\frac{x}{n\pi}\right)^{2k+1}\right).
\end{aligned}$$

If we change the order of summation and replace k with $k+1$, we get

$$\begin{aligned}
\cot x &= \frac{1}{x} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{2x^{2k-1}}{(n\pi)^{2k}}\right) \\
&= \frac{1}{x} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{2x^{2k-1}}{(\pi)^{2k}} \cdot \frac{1}{n^{2k}} \\
&= \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x^{2k-1}}{(\pi)^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} \\
&= \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x^{2k-1}}{(\pi)^{2k}}\zeta(2n).
\end{aligned}$$

Now, we have two different equations for cotangent: one which involves the Bernoulli numbers and another that involves the even values of the zeta function. We are finally ready to prove Theorem 6.2.

Proof. Recall that the equation for cotangent from section 5 was

$$\cot = \sum_{k=0}^{\infty}(-1)^k\frac{2B_{2k}(2x)^{2k-1}}{(2k)!} = \frac{1}{x} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2B_{2k}(2x)^{2k-1}}{(2n)!}. \tag{3}$$

If we equate these expressions, we find

$$\begin{aligned}
\frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x^{2k-1}}{(\pi)^{2k}}\zeta(2k) &= \frac{1}{x} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2B_{2k}(2x)^{2k-1}}{(2k)!} \\
\sum_{k=1}^{\infty}\frac{2x^{2k-1}}{(\pi)^{2k}}\zeta(2n) &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2B_{2k}(2x)^{2k-1}}{(2k)!}.
\end{aligned}$$

When we compare the coefficients of x^{2k+1} in the two summations, we arrive at the equation

$$\frac{2}{\pi^{2k}}\zeta(2k) = (-1)^{k-1}\frac{2^{2k}B_{2k}}{(2k)!}$$

which we can rearrange as a formula for $\zeta(2k)$:

$$\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}}{(2k)!}B_{2k}.$$

□

Using cotangent as a bridge, we have uncovered a result that some mathematicians consider one of Euler's most astounding [6]. An immediate corollary is one of our preliminary observations: the even Bernoulli numbers alternate sign. We notice that all $\zeta(2k)$ must always be positive, so from the exponent of -1, we have $B_{4k} < 0$ and $B_{4k+2} > 0$ for integers k. Thus, another way to write this result above, recognizing that $(-1)^{k-1}B_{2k} = |B_{2k}|$, is:

$$\zeta(2k) = \frac{|B_{2k}|(2\pi)^{2k}}{(2k)!}.$$

We also get the following important result about the positive growth of the Bernoulli numbers:

Corollary 6.2.1. *For $k \geq 3$, $|B_{2k+2}| > |B_{2k}|$*

Looking at the first few terms, we might not make this conjecture, since $|B_6| = \frac{1}{42} < \frac{1}{30} = |B_4|$. However, for all other k this is true.

Proof. Using the above formula, we solve for the magnitude of two consecutive even Bernoulli numbers:

$$|B_{2k}| = \frac{2\zeta(2k)(2k)!}{(2\pi)^{2k}} \qquad |B_{2k+2}| = \frac{2\zeta(2k+2)(2k+2)!}{(2\pi)^{2k+2}}.$$

If we take the ratio of these terms, we find

$$\begin{aligned}\frac{|B_{2k+2}|}{|B_{2k}|} &= \frac{(2k+2)!(2\pi)^{2k}}{(2m)!(2\pi)^{2m+2}} \\ &= \frac{(2m+2)(2m+1)}{(2\pi)^2} \\ &> 1\end{aligned}$$

for all $m \geq 3$. ☐

Now that we have proven these useful corollaries about the Bernoulli numbers, let us consider some examples that apply this formula to mathematical problems.

Example 6.1. Evaluate the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{1}{1^{10}} + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \cdots$$

Solution. We recognize that this sum is $\zeta(10)$, and we can use the formula derived above.

$$\begin{aligned}\zeta(10) &= \frac{4^5|B_{10}|\pi^{10}}{10!} \\ &= \frac{1024|\frac{5}{66}|\pi^{10}}{10!} \\ &= \frac{1}{93555}\pi^{10} \\ &\approx 1.0009945.\end{aligned}$$

The value of $\frac{1}{1^{10}} + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^{10}}$ is very close to 1. ☐

This calculation is complex enough with computers available, but when Euler discovered this formula, he made his calculations manually. Despite the computational difficulty, he was able to find even values of the zeta function up to $\zeta(26)$.

Example 6.2. For each positive N, let P_N be the probability that two randomly chosen integers in $\{1, 2, \ldots, N\}$ are coprime. As N approaches infinity, to what value P does P_N converge?

Solution. We claim that

$$P_N \to P = \frac{6}{\pi^2}.$$

We will "prove" this in similar fashion to our other Euler proof in this section, that is, without the detailed consideration of convergence that we would need in a rigorous solution.

We consider each of the possible prime factors of the integers below N. There is one-half probability that each of the integers are even (or nearly one-half, depending on the value of N), so the probability that they both are even is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Thus, the probability that the integers do not share 2 as a factor is exactly $(1 - \frac{1}{4})$.

Applying similar logic convinces us that the probability that the integers do not share a factor of 3 is $(1 - \frac{1}{9})$, that they do not share a factor of 4 is $(1 - \frac{1}{16})$, and so on, so that the probability that any prime p is not a common factor is $(1 - \frac{1}{p^2})$. Thus, with an error term of ϵ_N, our desired probability becomes

$$\begin{aligned} P_N &= \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots + \epsilon_N \\ &= \prod_{\text{prime } p} \left(1 - \frac{1}{p^2}\right) + \epsilon_N. \end{aligned}$$

Here, we will wave our hands. The formal result, which shows that the error ϵ_N converges to 0, was proven in 1881 by Italian mathematician Ernesto Césaro. We will take this for granted and state that

$$\lim_{N\to\infty} P_N = P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots$$

We now draw upon Proposition 6.1 to write P in terms of the zeta function

$$P = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 61\%$$

so there is a little less than two-thirds chance that the numbers will be coprime. □

Example 6.3. If we think about it closely, we can actually combine the results of the previous two examples. Select *ten* integers between 1 and N at random and let the probability that they are collectively coprime be P_N. Then as N approaches infinity, what is the value of *this* probability P?

We build upon the logic of the previous example. The probability that each number has a factor of 2 is nearly $1 - \frac{1}{2^{10}}$. The probability that each has a factor of 3 is about $1 - \frac{1}{3^{10}}$, and so on. Then,

$$\begin{aligned} \lim_{N\to\infty} P_N = P &= \left(1 - \frac{1}{2^{10}}\right)\left(1 - \frac{1}{3^{10}}\right)\left(1 - \frac{1}{5^{10}}\right) \cdots \\ &= 1/\zeta(10) \\ &\approx 99.9\% \end{aligned}$$

so it is almost certain that the ten numbers will not all share a common factor.

These applications are fascinating; however, they are not the primary use of the zeta function. The Riemann zeta function is more famous as a complex function, with powers k in the complex plane. In 1859, Bernhard Riemann hypothesized a result related to the complex Riemann zeta function, namely, that all of its nontrivial zeroes lie on the line $x = \frac{1}{2}$. The conjecture has never been proven and remains one of the great unsolved problems of mathematics. Mathematicians and mathematical physicists have developed a whole branch of mathematics contingent on the fact that the hypothesis is true, so that anyone who manages to uncover the proof will immediately verify thousands of results. The interested reader may want to read about this conjecture and the complex zeta function online, or watch the explanatory video of 3Blue1Brown on YouTube.

7 Bernoulli Polynomials

The Bernoulli polynomials are a generalization of the Bernoulli numbers. They have a variety of interesting properties, and will feature in our proof of the Euler-Maclaurin Summation Formula.

Definition 7.1. The *Bernoulli polynomials* are a sequence of polynomials, $B_k(y)$, defined by the following power series expansion:

$$\frac{xe^{xy}}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k(y)x^k}{k!}.$$

The generating function for the Bernoulli polynomials is the generating function for the Bernoulli numbers multiplied by a term of e^{xy}. Our first observation of the Bernoulli polynomials is that the constant term of $B_k(y)$ is in fact B_k. If we set $y = 0$, then $\frac{xe^{xy}}{e^x-1} = \frac{x}{e^x-1}$ and so $B_k(0) = B_k$.

There are further connections: we can use the generating function for the Bernoulli numbers to develop a recurrence relation for the Bernoulli polynomials.

Proposition 7.1. The Bernoulli polynomials, $B_k(y)$ satisfy the recurrence relation

$$B_k(y) = \sum_{n=0}^{k} \binom{k}{n} B_n y^{k-n}.$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{B_k(y)x^k}{k!} &= \frac{xe^{xy}}{e^x - 1} \\
&= \frac{x}{e^x - 1} \cdot e^{xy} \\
&= \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} \cdot \sum_{k=0}^{\infty} \frac{(xy)^k}{k!}.
\end{aligned}$$

We find the Cauchy product by the same process as earlier to obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{B_k(y)x^k}{k!} &= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{(xy)^{k-n}}{(k-n)!} \cdot \frac{B_n x^n}{n!} \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{y^{k-n}B_k}{(k-n)!n!}x^k \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \binom{k}{n} y^{k-n} B_n \frac{x^k}{k!}.\end{aligned}$$

Now, if we compare terms of x in the right and left-hand summations, we see the following:

$$B_k(y) = \sum_{n=0}^{k} \binom{k}{n} B_n y^{k-n}$$

as desired. □

Using this recurrence, we can calculate the first few Bernoulli polynomials:

$$\begin{aligned}B_0(y) &= 1 \\ B_1(y) &= y - \frac{1}{2} \\ B_2(y) &= y^2 - y + \frac{1}{6} \\ B_3(y) &= y^3 - \frac{3}{2}y^2 + \frac{1}{2}y \\ B_4(y) &= y^4 - 2y^3 + y^2 - \frac{1}{30} \\ B_5(y) &= y^5 - \frac{5}{2}y^4 + \frac{5}{3}y^3 - \frac{1}{6}y \\ B_6(y) &= y^6 - 3y^5 + \frac{5}{2}y^4 - \frac{1}{2}y^2 + \frac{1}{42}.\end{aligned}$$

Notice our earlier result that the constant term of each polynomial is a Bernoulli number.

Let us consider one of these polynomials, say $B_5(y)$, more closely. What if we differentiate it, or integrate it over 0 to 1?

$$\begin{aligned}\frac{d}{dy}B_5(y) &= 5y^4 - 10y^3 + 5y^2 - \frac{1}{6} \\ &= 5(y^4 - 2y^3 + y^2 - \frac{1}{30}) \\ &= 5B_4(y)\end{aligned}$$

so we observe that in this case, $B_k'(y) = kB_{k-1}(y)$. We also see

$$\begin{aligned}\int_0^1 B_5(y) &= \frac{1}{6}y^6 - \frac{1}{2}y^5 + \frac{5}{12}y^4 - \frac{1}{12}y^2\Big|_0^1 \\ &= 0.\end{aligned}$$

These two observations are, remarkably, true in general. In fact, they give us the following inductive definition for the Bernoulli numbers.

Proposition 7.2. [16] A polynomial, $B_k(y)$, is a *Bernoulli polynomial* if and only if

1. $B_0(y) = 1$;
2. $B'_k(y) = kB_{k-1}(y)$;
3. $\int_0^1 B_k(y)dy = 0$ for $k \geq 1$.

Are these polynomials and those we found with the generating function $\frac{xe^{xy}}{e^x-1}$ exactly the same? They are. To prove this, we first demonstrate this recursive sequence of polynomials is unique. Then, we prove our original definition of the Bernoulli polynomials satisfies these three properties. Therefore, the two definitions are equivalent.

Our new inductive definition generates a unique sequence. The first property specifies the first term. The second property determines each next term up to a constant. The third property determines exactly that constant. Therefore, the sequence of polynomials must be unique.

We also show the original definition of the Bernoulli polynomials fulfills each of these properties. By definition, the first property is satisfied. For the second property, we take our generating function for the Bernoulli polynomials and differentiate with respect to y.

$$\begin{aligned} \frac{d}{dy}\frac{xe^{xy}}{e^x-1} &= \frac{d}{dy}\sum_{k=0}^{\infty}\frac{B_k(y)x^k}{k!} \\ \frac{x^2e^{xy}}{e^x-1} &= \sum_{k=1}^{\infty}\frac{B'_k(y)x^k}{k!}. \end{aligned}$$

Next, we divide both sides by x and then replace k with $k+1$:

$$\begin{aligned} \frac{xe^{xy}}{e^x-1} &= \sum_{k=1}^{\infty}\frac{B'_k(y)x^{k-1}}{k!} \\ &= \sum_{k=0}^{\infty}\frac{B'_{k+1}(y)x^k}{(k+1)!}. \end{aligned}$$

Finally, we equate terms of the two expansions of $\frac{xe^{xy}}{e^x-1}$ and find

$$\begin{aligned} \frac{B'_{k+1}(y)}{(k+1)!} &= \frac{B_k(y)}{k!} \\ B'_{k+1}(y) &= (k+1)B_k(y). \end{aligned}$$

The third property is also satisfied,

$$\begin{aligned} \int_0^1 B_k(y)dy &= \left.\frac{B_{k+1}(y)}{k+1}\right|_0^1 \\ &= \frac{1}{k+1}(B_{k+1}(1) - B_{k+1}(0)) \\ &= 0 \end{aligned}$$

because $B_{k+1}(0) = B_{k+1}$ and $B_{k+1}(1) = \sum_{n=0}^{k+1} \binom{k+1}{n} B_n = B_{k+1}$. Therefore, our original definition of the Bernoulli polynomials satisfies these three properties.

As we mentioned earlier, the Bernoulli polynomials are a generalization of the Bernoulli numbers. The reader may be interested to know that the Bernoulli polynomials can be generalized even further.

Definition 7.2. The *generalized Bernoulli polynomials* $B_n^{(\alpha)}(x)$ are defined by the following generating function [22]:

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}.$$

We will encounter this definition later during our discussion of the *Bernoulli matrix* in section 13. For now, we simply note that $B_n^{(1)}(x) = B_n(x)$ and $B_n^{(1)}(0) = B_n$, the Bernoulli polynomials and Bernoulli numbers, respectively.

We end this section by proving three interesting properties (posed by [13]) about the Bernoulli polynomials, in order to convince the reader that this sequence of polynomials is particularly special.

1. $B_k(y+1) - B_k(y) = ky^{k-1}$.

Proof. We prove this identity by manipulating the following generating function:

$$\begin{aligned}
\sum_{k=0}^{\infty} (B_k(y+1) - B_k(y)) \frac{x^k}{k!} &= \sum_{k=0}^{\infty} B_k(y+1) \frac{x^k}{k!} - \sum_{k=0}^{\infty} B_k(y) \frac{x^k}{k!} \\
&= \frac{xe^{x(y+1)}}{e^x - 1} - \frac{xe^{xy}}{e^x - 1} \\
&= \frac{xe^{x(y+1)} - xe^{xy}}{e^x - 1} \\
&= \frac{xe^{xy}(e^x - 1)}{e^x - 1} \\
&= xe^{xy} \\
&= \sum_{k=0}^{\infty} \frac{x(xy)^k}{k!} \\
&= \sum_{k=0}^{\infty} y^k \frac{x^{k+1}}{k!} \\
&= \sum_{k=0}^{\infty} ky^k \frac{x^k}{k!}.
\end{aligned}$$

Comparing powers of x yields the identity. □

2. $B_k(1-y) = (-1)^k B_k(y)$.

Proof. We prove this and the following result using a similar technique:

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k(B_k(y))\frac{x^k}{k!} &= \sum_{k=0}^{\infty}(B_k(y))\frac{(-x)^k}{k!} \\ &= \frac{-xe^{-xy}}{e^{-x}-1} \\ &= \frac{-xe^{-xy}}{e^{-x}-1}\left(\frac{-e^x}{-e^x}\right) \\ &= \frac{xe^{-xy}e^x}{e^x-1} \\ &= \frac{xe^{x(1-y)}}{e^x-1} \\ &= \sum_{k=0}^{\infty}(B_k(1-y))\frac{x^k}{k!}.\end{aligned}$$

A comparison of powers again gives us the identity. □

3. $B_k(\frac{1}{2}) = (2^{1-k}-1)B_k$

   Proof.

$$\begin{aligned}\sum_{k=0}^{\infty}(2^{1-k}-1)B_k\frac{x^k}{k!} &= \sum_{k=0}^{\infty}2^{1-k}B_k\frac{x^k}{k!} + \sum_{k=0}^{\infty}B_k\frac{x^k}{k!} \\ &= 2\sum_{k=0}^{\infty}B_k\frac{(x/2)^k}{k!} + \sum_{k=0}^{\infty}B_k\frac{x^k}{k!} \\ &= 2\frac{x/2}{e^{x/2}-1} - \frac{x}{e^x-1} \\ &= \frac{x(e^{x/2}+1)-x}{e^x-1} \\ &= \frac{xe^{x/2}}{e^x-1} \\ &= \sum_{k=0}^{\infty}B_k\left(\frac{1}{2}\right)\frac{x^k}{k!}.\end{aligned}$$

   Comparison of powers yields the desired identity. □

There are several other properties of the Bernoulli polynomials we encourage the reader to prove [18]:

1. $B_k(1) = (-1)^k B_k$
2. $B_{2k}(\frac{1}{3}) = -\frac{1}{2}(1-3^{1-2k})B_{2k}$
3. $B_{2k}(\frac{1}{6}) = \frac{1}{2}(1-2^{1-2k})(1-3^{1-2k})B_{2k}$
4. $B_k(y) = (B+y)^k$, where B^k is interpreted as the Bernoulli number B_k (much like the mnemonic for the Bernoulli numbers themselves)
5. $B_k = (B-y)^k$, where B_k is interpreted as the Bernoulli *polynomial* $B_k(y)$.

The properties make the Bernoulli polynomials a valuable analytic tool. Armed with this useful extension of the Bernoulli numbers, we can fully grasp our next result, which ties together integration and summation in one powerful formula.

8 The Euler-Maclaurin Summation Formula

One of the most useful results involving the Bernoulli numbers is a formula connecting summations and integrals discovered independently by Euler and the Scottish mathematician Colin Maclaurin (1698-1746), called the Euler-Maclaurin Summation Formula (EMSF).

Theorem 8.1. *Let a and b be integers with $a < b$ and let f be a smooth function on $[a, b]$. Then for all $m \geq 1$:*

$$\sum_{i=a}^{b-1} f(i) = \int_a^b f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_a^b + R_m$$

where R_m, the remainder, is equal to $(-1)^{m+1} \int_a^b \frac{B_m(y-\lfloor y \rfloor)}{m!} f^{(m)}(x)dx$ and tends towards zero as m approaches infinity.

The EMSF is quite the expression to unpack. But the key idea is that given a sufficiently nice function f (differentiable $m-1$ times), we can write a summation as the sum of an integral, a term involving Bernoulli numbers, and a remainder. The next section will explore the various ways to apply this powerful formula. First, we will prove it.

Proof. (We follow closely the proof from [3]) We proceed by induction on m and a summation. We will consider only $f(0)$ on the left hand side and show the result is true for all m. Later, we will show how this method works for all possible a and b. Our first step will be to prove the base case of $a = 0$, $b = 1$, and $m = 1$. That is,

$$f(0) = \int_0^1 f(x)dx + B_1 f(x)\Big|_0^1 + R_m \tag{4}$$

where $R_m = \int_0^1 B_1(x)f'(x)dx$.

We begin with a result from the Fundamental Theorem of Calculus,

$$f(x) = f(0) + \int_0^x f'(t)dt$$

If we integrate with respect to x over the interval 0 to 1, we see that

$$\begin{aligned}\int_0^1 f(x)dx &= \int_0^1 f(0) + \int_0^x f'(t)dtdx \\ &= f(0) + \int_0^1 \int_0^x f'(t)dtdx.\end{aligned}$$

We switch the order of integration, so that x ranges from t to 1 as t ranges from 0 to 1.

$$\begin{aligned}
\int_0^1 f(x)dx &= f(0) + \int_0^1 \int_t^1 f'(t)dxdt \\
&= f(0) + \int_0^1 f'(t)(1-t)dt \qquad (5) \\
&= f(0) + \int_0^1 f'(t)dt + \int_0^1 f'(t)(-t)dt \\
&= f(0) + [f(1) - f(0)] + \int_0^1 f'(t)(-t)dt \\
&= f(1) + \int_0^1 f'(t)(-t)dt. \qquad (6)
\end{aligned}$$

Next, we add together equations 5 and 6.

$$\begin{aligned}
2\int_0^1 f(x)dx &= f(0) + f(1) + \int_0^1 f'(t)(1-t)dt + \int_0^1 f'(t)(-t)dt \\
2\int_0^1 f(x)dx &= f(0) + f(1) + \int_0^1 f'(t)(1-2t)dt.
\end{aligned}$$

If we divide by 2 and bring a term of $f(0)$ to the left side, we obtain

$$\begin{aligned}
\int_0^1 f(x)dx &= \frac{f(0)+f(1)}{2} + \int_0^1 f'(t)(\frac{1}{2} - t)dt \\
\int_0^1 f(x)dx &= \frac{f(1)-f(0)}{2} + f(0) + \int_0^1 f'(x)(\frac{1}{2} - x)dt \\
f(0) &= \int_0^1 f(x)dx - \frac{1}{2}(f(1) - f(0)) - \int_0^1 f'(x)(\frac{1}{2} - x) \\
f(0) &= \int_0^1 f(x)dx + B_1 f(x)\Big|_0^1 + R_m
\end{aligned}$$

with $R_m = \int_0^1 B_1(x) f'(x)dx$. This is our desired base case (equation 5). Next, we complete our induction step. Holding $a = 0$ and $b = 1$ constant, we show that for all $m \geq 1$

$$f(0) = \int_0^1 f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_0^1 + R_m \qquad (7)$$

with the remainder $R_m = (-1)^{m+1} \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)dx$. Assume that the above is true for all $k \leq m$. We turn our focus to the remainder term,

$$\frac{(-1)^{m+1}}{m!} \int_0^1 B_m(x) f^{(m)}(x)dx.$$

How can we evaluate this? Our key is to use the fact that $B'_{k+1}(x) = (k+1)B_k(x)$. We consider only

$$\int_0^1 B_m(x) f^{(m)}(x)dx.$$

and conduct integration by parts, which yields:

$$\int_0^1 B_k(x)f^{(k)}dx = \frac{B_{k+1}}{k+1}f^{(k)}(x)\Big|_0^1 - \frac{1}{k+1}\int_0^1 B_{k+1}(x)f^{(k+1)}(x)dx.$$

Substituting this in, the remainder term becomes

$$\begin{aligned}
R_m &= \frac{(-1)^{m+1}}{m!}\int_0^1 B_m(x)f^{(m)}(x)dx \\
&= \frac{(-1)^{m+1}}{m!}\left[\frac{B_{m+1}(x)}{m+1}f^{(m)}(x)\Big|_0^1 - \frac{1}{m+1}\int_0^1 B_{m+1}(x)f^{(m+1)}(x)dx\right] \\
&= \frac{(-1)^{m+1}}{(m+1)!}\left[B_{m+1}(x)f^{(m)}(x)\Big|_0^1 - \int_0^1 B_{m+1}(x)f^{(m+1)}(x)dx\right].
\end{aligned}$$

Here, we note if m is odd, then $(-1)^{m+1} = 1$. If m is even, then $B_{m+1}(0) = B_{m+1}(1) = 0$, from the previous section, and $B_{m+1} = 0$ because it is an odd Bernoulli number greater than 1. Therefore, we claim

$$\frac{(-1)^{m+1}}{(m+1)!}B_{m+1}(x)f^{(m)}(x)\Big|_0^1 = \frac{1}{(m+1)!}B_{m+1}f^{(m)}(x)\Big|_0^1$$

and the value of $f(0)$ becomes

$$\begin{aligned}
f(0) &= \int_0^1 f(x)dx + \sum_{k=1}^{m}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_0^1 + R_m \\
&= \int_0^1 f(x)dx + \sum_{k=1}^{m}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_0^1 + \frac{(-1)^{m+1}}{(m+1)!}\left[B_{m+1}(x)f^{(m)}(x)\Big|_0^1 - \int_0^1 B_{m+1}(x)f^{(m+1)}(x)dx\right] \\
&= \int_0^1 f(x)dx + \sum_{k=1}^{m}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_0^1 + \frac{1}{(m+1)!}B_{m+1}f^{(m)}(x)\Big|_0^1 - \frac{(-1)^{m+2}}{(m+1)!}\int_0^1 B_{m+1}(x)f^{(m+1)}(x)dx \\
&= \int_0^1 f(x)dx + \sum_{k=1}^{m+1}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_0^1 + \frac{(-1)^{m+1}}{(m+1)!}\int_0^1 B_{m+1}(x)f^{(m+1)}(x)dx.
\end{aligned}$$

This completes this round of induction.

In the second step of this proof, we take $f(i+x)$ for each integer $a \le i < b$. We sum,

$$\begin{aligned}
\sum_{i=a}^{b-1} f(i) &= \sum_{i=a}^{b-1}\left[\int_i^{i+1} f(x)dx + \sum_{k=1}^{m+1}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_i^{i+1} + \frac{(-1)^{m+1}}{(m+1)!}\int_i^{i+1} B_{m+1}(x)f^{(m+1)}(x)dx\right] \\
&= \sum_{i=a}^{b-1}\left[\int_i^{i+1} f(x)dx\right] + \sum_{i=a}^{b-1}\left[\sum_{k=1}^{m+1}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_i^{i+1}\right] + \sum_{i=a}^{b-1}\left[\frac{(-1)^{m+1}}{(m+1)!}\int_i^{i+1} B_{m+1}(x)f^{(m+1)}(x)dx\right] \\
&= \int_a^b f(x)dx + \sum_{k=1}^{m+1}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_a^b + \frac{(-1)^{m+1}}{(m+1)!}\int_a^b B_{m+1}(x)f^{(m+1)}(x)dx.
\end{aligned}$$

This is the Euler-Maclaurin Summation Formula. □

9 Applications of Euler-Maclaurin Summation

What is the significance of this statement? On one hand, the EMSF allows us to make close approximations by calculating sums in terms of integrals and vice versa. For example, we can consider the Riemann zeta function. While we have already derived an explicit formula for the even terms of the zeta function, the EMSF allows us to accurately approximate odd terms. The value of $\zeta(3)$ may not be expressible using any notation we currently have; however, using the EMSF we can approximate it to be 1.20205.

On the other hand, the formula allows us to do more than approximate. We can use it to prove a variety of important concrete results, three of which we discuss in the following subsections.

9.1 Revisiting the Sums of Powers

As promised earlier, the EMSF provides a simple proof for the formula for the sums of powers. As we conjectured in equation 1,

Corollary 9.0.1. *The sum of the first $n-1$ positive integers to the m^{th} power is equivalent to*

$$S_m(n-1) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m-k+1}. \tag{8}$$

Proof. We will apply the EMSF to the function $f(x) = x^p$, with $a = 0, b = n$ and $m \geq 1$. We consider the remainder first. For all $p \leq m$,

$$f^{(p)}(x) = m(m-1)(m-2)\cdots(m-p+1)x^{m-p}.$$

Therefore, $f^{(m)}(x) = m!$ and the remainder becomes

$$\begin{aligned}
R_m &= \frac{(-1)^{m+1}}{m!}\int_a^b B(y-\lfloor y\rfloor)f^{(m)}dy \\
&= (-1)^{m+1}\int_a^b B(y-\lfloor y\rfloor)dy \\
&= (-1)^{m+1}\left(\int_a^{a+1} B(y-\lfloor y\rfloor)dy + \int_{a+1}^{a+2} B(y-\lfloor y\rfloor)dy + \cdots + \int_{b-1}^{b} B(y-\lfloor y\rfloor)dy\right) \\
&= (-1)^{m+1}\left(\int_0^1 B(y-\lfloor y\rfloor)dy + \int_0^1 B(y-\lfloor y\rfloor)dy + \cdots + \int_0^1 B(y-\lfloor y\rfloor)dy\right) \\
&= (-1)^{m+1}(b-a)\int_0^1 B(y-\lfloor y\rfloor)dy \\
&= 0.
\end{aligned}$$

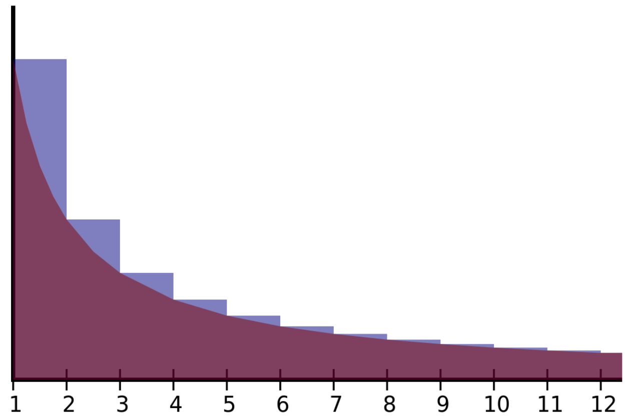

Figure 3: The Euler-Mascheroni constant is the convergent sum of the difference between $\{\frac{1}{x}\}$ and $\frac{1}{x}$

Thus, EMSF gives us

$$\begin{aligned}\sum_{i=0}^{n-1} x^m &= \int_0^n x^m + \sum_{k=1}^{m} \frac{B_k}{k!} m(m-1)(m-2)\cdots(m-k+2)x^{m-k+1}\Big|_0^n + R_m \\ &= \frac{x^{m+1}}{m+1} + \frac{1}{m+1}\sum_{k=1}^{m}\binom{m+1}{k} B_k n^{m-k+1} \\ &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m-k+1}\end{aligned}$$

which is the formula for the sums of powers that Bernoulli observed. □

There are other proofs of this fact, including induction proofs [6]; however, this is among the most elegant.

9.2 Euler's Constant

A second application of the EMSF gives us an expression for the *Euler constant* in terms of the Bernoulli numbers.

Definition 9.1. The *Euler-Mascheroni constant*, γ, is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \int_1^{\infty}\frac{1}{x}\right) \approx 0.57721566.$$

In other words, it is the value to which the difference between the harmonic series $\{\frac{1}{x}\}$ and the function $f(x) = \frac{1}{x}$ converges as x approaches infinity. It is found in many results in calculus, although it is not even known if the constant is irrational. In fact, it is allegedly the case that G. H. Hardy offered to give up his Savilian Chair at Oxford to the mathematician who confirmed the irrationality of the constant [18].

Using our summation formula, we uncover a the following relation between the Bernoulli numbers and the Euler-Mascheroni constant:

Corollary 9.0.2.

$$\gamma = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k}$$

Proof. With the EMSF, let $f(x) = \frac{1}{x}$, $a = 1$, and $b = n$.

$$\sum_{i=a}^{b-1} f(i) = \int_a^b f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\bigg|_a^b + R_m$$
$$\sum_{i=1}^{n-1} \frac{1}{i} = \int_1^n \frac{1}{x}dx + \sum_{k=1}^{m} \frac{B_k}{k!} \frac{(-1)^{k-1}(k-1)!}{x^k}\bigg|_1^n + R_m.$$

We evaluate the integral

$$\sum_{i=1}^{n-1} \frac{1}{i} = \ln x + \sum_{k=1}^{m} \frac{(-1)^{k-1}B_k}{kx^k}\bigg|_1^n + R_m$$
$$\sum_{i=1}^{n} \frac{1}{i} - \ln x = \frac{1}{n} + \sum_{k=1}^{m} \frac{(-1)^{k-1}B_k}{k}\left[\frac{1}{n^k} - 1\right] + R_m$$
$$= \frac{1}{2} + \frac{1}{2n} + \sum_{k=1}^{m} \frac{B_{2k}}{2k}\left[1 - \frac{1}{n^{2k}}\right] + R_m$$

since all odd terms of the Bernoulli numbers aside from B_1 are zero. If we take these sums as n approaches infinity, the left-hand side becomes the Euler-Mascheroni constant. And as m approaches infinity, we know that R_m goes to zero, giving us:

$$\lim_{m,n\to\infty}\left[\sum_{i=1}^{n} \frac{1}{i} - \ln x\right] = \lim_{m,n\to\infty}\left[\frac{1}{2} + \frac{1}{2n} + \sum_{k=1}^{m} \frac{B_{2k}}{2k}\left[1 - \frac{1}{n^{2k}}\right] + R_m\right]$$
$$\gamma = \frac{1}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{2k}.$$

□

After completing this proof, one might wonder if this same process could be used on the more general $f(x) = \frac{1}{x^k}$. It can, and although we will not examine this result here, it leads to an expression for the Riemann zeta function which is equivalent to the one we derived in section 6.

9.3 Stirling's Formula

A third major application of the EMSF is in the proof of Stirling's formula for the approximation of factorials. The factorial, $n!$, is one of the most common operations in mathematics, but also difficult to calculate for very large integers n. The Scottish mathematician James Stirling (1692-1770) was the one determine a useful approximation, using a result derived using the Bernoulli numbers.

Corollary 9.0.3. *For positive integers n,*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Proof. Let $f(x) = \ln n$, $a = 1$, $b = n$, and $m = 1$ in the EMSF.

$$\begin{aligned}
\sum_{i=a}^{b-1} f(i) &= \int_a^b f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_a^b + R_m \\
\sum_{i=1}^{n-1} \ln i &= \int_1^n \ln x dx + B_1 \ln x\Big|_1^n + R_1 \\
\ln(n-1)! &= n\ln n - n + 1 - \frac{1}{2}\ln n + R_1 \\
\ln n! &= n\ln n - n + 1 + \frac{1}{2}\ln n + R_1 \\
\ln n! &= (n + \frac{1}{2})\ln n - (n-1) + R_1 \\
n! &= C(n)\sqrt{n}\left(\frac{n}{e}\right)^n
\end{aligned}$$

where $C(n) = e^{R_1 - 1}$ and $\lim_{n\to\infty} C(n) = \sqrt{2\pi}$. So, $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$. □

Much like the Bernoulli numbers themselves, this result may be misattributed. The approximation bears Sitrling's name, but historians note that is was French mathematician de Moirve who developed the idea and completed most of the proof in 1733. Stirling's improvement—which some claim is the most critical component of the theorem and deserving of primary recognition—was the identification of the $\sqrt{2\pi}$ constant term [22]. While Stirling's proof is not included in this primer, it relies on a fascinating result of English mathematician John Wallis (1616-1703) that is worth mentioning:

Proposition 9.1.

$$\lim_{n\to\infty} \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)}\frac{1}{\sqrt{n}} = \sqrt{\pi}.$$

These proofs are a sampling of the applications of the EMSF. Beyond these, the formula is useful for all manner of approximations and derivations. As we will find, Stirling's Formula also helps us to circle back and examine the Bernoulli numbers again.

10 The Bernoulli Numbers Grow Large

In section 6, we found an expression for the even values of the Riemann zeta function. In the previous section, we discussed how the EMSF can help prove Stirling's Formula. If we combine these two, we find the following straightforward method to approximate Bernoulli numbers and understand just how quickly they grow.

Proposition 10.1. [7] For positive values of k,

$$B_{2k} \sim (-1)^{k-1} 4\left(\frac{k}{\pi e}\right)^{2k}\sqrt{\pi k}.$$

Proof. We know from section 6 that

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}.$$

We can rearrange this equation to find an expression in terms of B_{2k},

$$B_{2k} = \frac{(-1)^{k-1}2(2k)!}{(2\pi)^{2k}}\zeta(2k).$$

If we take the limit

$$\begin{aligned}\lim_{k\to\infty} \zeta(2k) &= \lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \\ &= 1 + \lim_{k\to\infty} \sum_{n=2}^{\infty} \frac{1}{n^{2k}} \\ &= 1\end{aligned}$$

so $\zeta(2k) \sim 1$ as k becomes large. With Stirling's Formula, we have

$$(2k)! \sim \left(\frac{2k}{e}\right)^{2k} \sqrt{4\pi k}.$$

Combining these observations, we see

$$B_{2k} \sim \frac{(-1)^{k-1} \cdot 2}{(2\pi)^{2k}} \left(\frac{2k}{e}\right)^{2k} \sqrt{4\pi k}$$

which turns out to be

$$B_{2k} \sim (-1)^{k-1} 4 \left(\frac{k}{\pi e}\right)^{2k} \sqrt{\pi k}$$

as desired. □

We can consider an alternative approximation. If we want to approximate the magnitude of the even Bernoulli numbers, we need only to take the absolute value:

$$|B_{2k}| \sim 4 \left(\frac{k}{\pi e}\right)^{2k} \sqrt{\pi k}.$$

There are other, more sophisticated approximations of the Bernoulli numbers that are used in computational studies. For example, from [6], we have fine-tuned upper and lower bounds.

Proposition 10.2.

$$4\left(\frac{2n}{4e\pi} \cdot \frac{120n^2+9}{120n^2-1}\right)^{n} \sqrt{\frac{\pi n}{2}} \le |B_n| \le 4\pi\left(\frac{2n+1}{4e\pi} \cdot \frac{240n(n+1)+69}{240n(n+1)+79}\right)^{n+1/2}$$

Now, let us ask ourselves a couple of questions related to the growth of the Bernoulli numbers.

Example 10.1. For what value of k does the value of B_{2k} first surpass one million?

Solution. We first need to solve the inequality

$$4\left(\frac{k}{\pi e}\right)^{2k}\sqrt{\pi k} > 1,000,000$$

and then show that the previous even valued Bernoulli number is less than one million. Solving for k, we find that it is between 12 and 13, so the nearest integer solution is $k = 13$. Plugging 13 into our approximation we get

$$\begin{aligned}|B_{26}| &\sim 4\left(\frac{13}{\pi e}\right)^{2\cdot 13}\sqrt{13\pi} \\ &\approx 1,420,955.\end{aligned}$$

Since this value is over one million, it is definitely a candidate! However, we need to make sure that the value of $|B_{24}|$ is not close to 1,000,000, which would force us to fine-tune our approximation. However,

$$\begin{aligned}|B_{24}| &\sim 4\left(\frac{12}{\pi e}\right)^{2\cdot 12}\sqrt{12\pi} \\ &\approx 86,280\end{aligned}$$

so our calculations produced the correct result. B_{26} is the first Bernoulli number with a magnitude greater than one million. □

Example 10.2. What about ten billion? What is the first Bernoulli number with magnitude greater than ten billion?

Solution. We solve

$$4\left(\frac{k}{\pi e}\right)^{2k}\sqrt{\pi k} > 10,000,000,000$$

and find that the first integer solution is $k = 16$. Using our approximations, we find

$$\begin{aligned}|B_{32}| &\sim 4\left(\frac{16}{\pi e}\right)^{2\cdot 16}\sqrt{16\pi} \\ &\approx 15,077,002,848\end{aligned}$$

and

$$\begin{aligned}|B_{30}| &\sim 4\left(\frac{15}{\pi e}\right)^{2\cdot 15}\sqrt{15\pi} \\ &\approx 599,912,195\end{aligned}$$

so B_{32} must be our desired Bernoulli number. □

Our next example is a useful and practical computational problem.

Example 10.3. Estimate B_{28} to the nearest integer.

Solution. For this problem, we will use the approximation from Proposition 10.2. Plugging in 28 for n, we can use a calculator to solve for the bounds of B_{28}:

$$4\left(\frac{2\cdot 28}{4e\pi}\cdot\frac{120\cdot 28^2+9}{120\cdot 28^2-1}\right)^{28}\sqrt{\frac{\pi\cdot 28}{2}}\leq |B_{28}|\leq 4\pi\left(\frac{2\cdot 28+1}{4e\pi}\cdot\frac{240\cdot 28(29)+69}{240\cdot 28(29)+79}\right)^{28+1/2}$$

which reduces to

$$27,298,230.96508\leq |B_{24}|\leq 27,298,230.96702.$$

Since $24=4\cdot 6$, then $B_{24}<0$. So the value of B_{24} to the the nearest integer is -27,298,231. □

While this method of estimation is computationally difficult, it is certainly faster than direct calculation using the recursive Bernoulli formula. For large numbers of n, this estimation becomes a particularly valuable tool in conjunction with the Clausen-von Staudt Theorem, as we will see in the next section.

11 The Clausen-von Staudt Theorem

This theorem was discovered independently by Thomas Clausen (1801-1885) and Karl von Staudt (1798-1867) in 1840. It allows one to easily compute Bernoulli numbers modulo 1. In effect, it gives the fractional part of a Bernoulli number, and consequently, its denominator.

Theorem 11.1. *For primes p,*

$$B_{2k}\equiv -\sum_{(p-1)|2k}\frac{1}{p} \mod 1$$

for primes p.

For example, with $k=10$, we know $p-1$ divides 20 for $p=2,3,5$, and 11, as $1,2,4$ and 10 divide 10.

$$\begin{aligned} B_{20} &\equiv -\sum_{(p-1)|2k}\frac{1}{p}\\ &= -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{11}\right)\\ &= -\frac{371}{330}\\ &\equiv \frac{289}{330} \mod 1. \end{aligned}$$

Observe that every Bernoulli denominator is the product of distinct primes.

One of the simplest consequences of the Clausen-von Staudt Theorem concerns primes of the form $k=3n+1$.

Corollary 11.1.1. *For prime $k=3n+1$,*

$$B_{2k}\equiv\frac{1}{6} \mod 1.$$

Proof. Consider the Clausen von Staudt theorem. If $(p-1)$ divides $2(3n+1)$, then $p-1$ can be $2,3,3n+1$, or $6n+2$. But if $p-1=k=3n+1$, a prime, then $p=3n+2$ must be divisible by 2, in which case p is not prime. Similarly, if $p-1 = 6n+2$, then $p = 3(2n+1)$, so p is not prime. Therefore, the only values of $p-1$ are 2 and 3, and

$$\begin{aligned} B_{3n+1} &\equiv -\sum_{(p-1)|2k} \frac{1}{p} \\ &= -\left(\frac{1}{2}+\frac{1}{3}\right) \\ &= -\frac{5}{6} \\ &\equiv \frac{1}{6} \mod 1. \end{aligned}$$

□

This gives us

$$B_{14} \equiv B_{26} \equiv B_{38} \equiv B_{62} \equiv B_{74} \equiv B_{86} \equiv B_{122} \equiv \cdots \equiv \frac{1}{6} \mod 1.$$

As we have shown, one reason why the Clausen-von Staudt Theorem is important because it allows us to calculate exactly the denominator of each Bernoulli number. No such analogue exists for the numerator. However, the theorem allows us to calculate the exact value of a Bernoulli number, as long as you have a close enough approximation.

Example 11.1. Find the exact value of B_{28} using approximation.

Solution. This problem is solved in two steps. First, we find an estimate of B_{28} to the nearest integer. As we solved in Example 10.3, $B_{28} \approx -27,298,230.96$.

Second, we use the Clausen-von Staudt to calculate $B_{28} \mod 1$. We know that $p-1$ divides 28 for $p = 2, 3, 5, 29$, since 1, 2, 4, and 28 divide 28. Thus,

$$\begin{aligned} B_{28} &\equiv -\sum_{(p-1)|2k} \frac{1}{p} \\ &= -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{29}\right) \\ &= -\frac{929}{870} \\ &= -\frac{59}{870} \mod 1. \end{aligned}$$

This is the key to solve for the Bernoulli number exactly. Since $\frac{811}{870} \approx .96$, the exact Bernoulli number B_{28} must be

$$\begin{aligned} B_{28} &= -27,298,231 - \frac{811}{870} \\ &= -\frac{23749460970 + 811}{870} \\ &= -\frac{23749461029}{870}. \end{aligned}$$

This result is correct [6]. So we have demonstrated a valuable way to calculate Bernoulli numbers with an accurate enough approximation. □

12 Direct Formulas

Approximation and the von Staudt theorem is one good way to get exact values for the Bernoulli numbers. Another method is through direct formulas. In this section we consider formulas that various mathematicians have proven to find Bernoulli numbers directly. The first we prove simply following a process from [8].

Proposition 12.1. The Bernoulli numbers can be written

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n.$$

Proof. We know that the Bernoulli numbers are defined

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

Then, by the definition of a Taylor series, B_n is the n^{th} derivative of $\frac{x}{e^x-1}$ evaluated at $x = 0$. In other words,

$$B_n = \frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) \bigg|_{x=0} \tag{9}$$

We have that $t = \ln(1 - (1 - e^t))$ with some algebraic manipulation. Since the Taylor series of $\ln(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k}$, we also have that $\ln(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$. Using these two pieces of information, we find:

$$x = \sum_{k=1}^{\infty} \frac{(1 - e^x)^k}{k}$$

for $|1 - e^x| < 1$. Thus, we can rewrite the generating function of the Bernoulli numbers,

$$\frac{x}{e^x - 1} = \sum_{k=1}^{\infty} \frac{(1 - e^x)^{k-1}}{k} \tag{10}$$

$$= \sum_{k=0}^{\infty} \frac{(1 - e^x)^k}{k+1} \tag{11}$$

with the replacement of k with $k + 1$. Since $\sum_{k=0}^{\infty} x^k/(k + 1)$ can be differentiated arbitrarily many times, we can substitute equation 10 into equation 9. Then,

$$B_n = \frac{d^n}{dx^n} \left(\sum_{k=0}^{\infty} \frac{(1 - e^x)^k}{k+1} \right) \bigg|_{x=0}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{d^n}{dx^n} (1 - e^x)^k \bigg|_{x=0}$$

We may notice that if $k \geq n + 1$, then the n^{th} derivative is zero. Then the binomial theorem gives us the expression,

$$B_n = \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{d^n}{dx^n} e^{jx} \bigg|_{x=0}$$

We evaluate the derivative to obtain

$$B_n = \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n$$

as desired. □

There are a variety of other formulas that mathematicians have discovered–and as H.W. Gould pointed out in a 1972 survey article–*rediscovered* over time. We encourage the interested reader to evaluate the first few terms of these two to get a feel for how the summations work:

1. $B_n = \frac{1}{n+1} \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^j j^n \frac{\binom{n+1}{k-j}}{\binom{n}{k}}$

2. $B_{2n} = \sum_{j=2}^{2n+!} (-1)^{j-1} \binom{2n+1}{j} \frac{1}{j} \sum_{k=1}^{j-1} k^{2n}$

These formulas, as well as several more, are documented in [11].

We will also briefly mention one other, *combinatorial*, definition. In a paper on what are called the *poly-Bernoulli numbers*, Brewbaker defines the Bernoulli numbers in terms of Sitrling numbers of the second kind [2].

Definition 12.1. The *Stirling number of the second kind*, $S(n, k)$, is the number of ways to partition n elements into k non-empty sets. The formula for this operation turns out to be

$$S(n,k) = \frac{(-1)^k}{k!} \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^n.$$

Using this definition, we can give the formula for the Bernoulli numbers:

$$B_n = \sum_{k=0}^{n} (-1)^{n+k} \frac{k! S(n,k)}{k+1}.$$

This formula provides a basis for an intriguing question. Do the Bernoulli numbers have a combinatorial interpretation? In other words, are the numerator and denominator of the numbers counting something?

While the Bernoulli numbers have been studied extensively, this is one possible area for growth in the literature. We conducted some investigation into this possibility, but were not able to make useful progress. Perhaps the motivated reader can pick up this open question and run with it.

13 Bernoulli Numbers in Matrices

In the last couple of sections of this primer, we examine a few applications of the Bernoulli numbers that may be relatively unexpected. Then we conclude, offering several directions in which the curious reader can continue to learn. In these subsections, we consider how the Bernoulli numbers can be derived from Pascal's matrices, find Bernoulli numbers in an interesting class of matrices, and then explore the applications of the so-called Bernoulli matrix.

13.1 Pascal's Matrix

As we have seen already, the Bernoulli recurrence relation $B_0 = 1, B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$ is closely related to Pascal's triangle. In this section we hope to solidify this connection with a derivation of the Bernoulli numbers that involves Pascal's matrices. We draw from the work of A.W.F. Edwards [10]. The first Pascal's matrix (in lower triangular form), as Edwards puts it, "can only be":

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . \\ 1 & 2 & 1 & 0 & 0 & . & . \\ 1 & 3 & 3 & 1 & 0 & . & . \\ 1 & 4 & 6 & 4 & 1 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

with the rule

$$P_{ij} = \begin{cases} \binom{i-1}{j-1} & i \geq j \\ 0 & i < j \end{cases}$$

The second Pascal matrix, Q, as Edwards defines it, is P without its main diagonal:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ 1 & 2 & 0 & 0 & 0 & . & . \\ 1 & 3 & 3 & 0 & 0 & . & . \\ 1 & 4 & 6 & 4 & 0 & . & . \\ 1 & 5 & 10 & 10 & 5 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

You may notice this looks like the recurrence relation for the Bernoulli numbers. This fact will prove critically useful. In 1654, Pascal derived a formula for $S_m(n)$ in terms of lower order sums of powers. Unlike a similar theorem of Pierre de Fermat, Pascal's method was more practical:

$$(n+1)^m = n + 1 + mS_1(n) + \binom{m}{2} S_2(n) + \binom{m}{3} + \cdots + \binom{m}{m-1} S_{m-1}(n)$$

If we write this as a series for linear equations for each positive integer m, we get the matrix equation

$$\begin{bmatrix} (n+1) \\ (n+1)^2 \\ (n+1)^3 \\ (n+1)^4 \\ (n+1)^5 \\ . \\ . \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ 1 & 2 & 0 & 0 & 0 & . & . \\ 1 & 3 & 3 & 0 & 0 & . & . \\ 1 & 4 & 6 & 4 & 0 & . & . \\ 1 & 5 & 10 & 10 & 5 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} n+1 \\ S_1(n) \\ S_2(n) \\ S_3(n) \\ S_4(n) \\ . \\ . \end{bmatrix}$$

If we invert Q and replace $n+1$ with n, we can multiply both sides on the left by Q^{-1} to obtain

$$\begin{bmatrix} n \\ S_1(n-1) \\ S_2(n-1) \\ S_3(n-1) \\ S_4(n-1) \\ . \\ . \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & . & . \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 & . & . \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & . & . \\ -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ . \\ . \end{bmatrix}$$

The first column of Q^{-1} is the sequence of the Bernoulli numbers. Why is that? Since $QQ^{-1} = I$, if we take Q multiplied by the first column vector of Q^{-1} we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ 1 & 2 & 0 & 0 & 0 & . & . \\ 1 & 3 & 3 & 0 & 0 & . & . \\ 1 & 4 & 6 & 4 & 0 & . & . \\ 1 & 5 & 10 & 10 & 5 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{6} \\ 0 \\ -\frac{1}{30} \\ . \\ . \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \end{bmatrix}$$

so the first column of Q^{-1} is simply the list of coefficients of B_k such that $\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$. As Edwards says, "How delighted Pascal must have been to learn that his own method for finding the sums of powers could be completed by inverting a matrix of coefficients of the Arithmetical Triangle!"

Also fascinating are the next rows of Q^{-1}. What are their significance? The second row of Q^{-1} gives us the following equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ 1 & 2 & 0 & 0 & 0 & . & . \\ 1 & 3 & 3 & 0 & 0 & . & . \\ 1 & 4 & 6 & 4 & 0 & . & . \\ 1 & 5 & 10 & 10 & 5 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{4} \\ 0 \\ . \\ . \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ . \\ . \end{bmatrix}$$

The terms of the second row of Q^{-1} give us the coefficients that satisfy $A_0 = 1, \sum_{k=1}^{n+1} \binom{n}{k} A_0$, a recurrence relation which selects all of the "inner," non-one elements Pascal's Triangle. It may seem this relation has little to do with anything, but who knows? It could be much like the Bernoulli numbers themselves: unexpectedly useful.

13.2 Bernoulli Numbers with Determinants

The Bernoulli numbers are also related to a group of determinants using factorials, "where the evaluation of these determinants by row and column manipulation is either quite challenging or almost impossible" [5]. Here, we derive an expression for the Bernoulli numbers in terms of a determinant, and allude to the uses for such an expression.

If we write the primary recurrence relation for the Bernoulli numbers as a system of equations

$$\begin{cases} 2B_1 + 1 = 0 \\ 3B_2 + 3B_1 + 1 = 0 \\ 4B_3 + 6B_2 + 4B_3 + 1 = 0 \\ 5B_4 + 10B_3 + 10B_2 + 5B_1 + 1 = 0 \\ \dots \\ \binom{n+1}{n} B_n + \cdots + \binom{n}{2} B_2 + \binom{n}{1} B_1 + 1 = 0 \end{cases}$$

then we can subtract 1,

$$\begin{cases} 2B_1 = -1 \\ 3B_2 + 3B_1 = -1 \\ 4B_3 + 6B_2 + 4B_3 = -1 \\ 5B_4 + 10B_3 + 10B_2 + 5B_1 = -1 \\ \dots \\ \binom{n+1}{n} B_n + \cdots + \binom{n}{2} B_2 + \binom{n}{1} B_1 = -1 \end{cases}$$

and replace $b_n = B_n/n!$ in the system,

$$\begin{cases} b_1 = -\frac{1}{2!} \\ b_2 + \frac{b_1}{2!} = -\frac{1}{3!} \\ b_3 + \frac{b_2}{2!} + \frac{b_3}{3!} = -\frac{1}{4!} \\ b_4 + \frac{b_3}{2!} + \frac{b_2}{3!} + \frac{b_1}{4!} = -\frac{-1}{5!} \\ \dots \\ b_n + \frac{b_{n-1}}{2!} + \cdots + \frac{b_2}{(n-1)!} + \frac{b_1}{n!} = -1 \end{cases}$$

By Cramer's Rule, we solve for b_n:

$$b_n = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & . & . & -\frac{1}{2!} \\ \frac{1}{2!} & 1 & 0 & 0 & 0 & . & . & -\frac{1}{3!} \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & . & . & -\frac{1}{4!} \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & . & . & -\frac{-1}{5!} \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & . & . & -\frac{1}{6!} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \frac{1}{(n-4)!} & . & . & -\frac{1}{(n+1)!} \end{vmatrix}$$

Using determinant properties, including factoring out a coefficient of -1, we get an interesting representation for the Bernoulli numbers:

$$B_n = n! \cdot b_n = \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & 0 & . & . & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & . & . & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & . & . & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \frac{1}{(n-4)!} & . & . & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & . & . & \frac{1}{2!} \end{vmatrix}$$

Using the equivalences about Bernoulli numbers that we already know, it is possible to use this representation to solve for determinants that would be very difficult by any conventional means. For several interesting problems, consult [5].

13.3 The Bernoulli Matrix

One generalization of the Bernoulli numbers yields what is known as the *Bernoulli matrix*, $\mathscr{B}$, defined at the $(i, j)^{\text{th}}$ entry by:

$$B_{i,j} = \begin{cases} \binom{i}{j} B_{i-j} & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases}$$

In matrix form, it looks like:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ \binom{2}{1}\frac{1}{2} & 1 & 0 & 0 & 0 & . & . \\ 0 & \binom{3}{2}\frac{1}{2} & 1 & 0 & 0 & . & . \\ \binom{4}{1}\frac{1}{6} & 0 & \binom{4}{3}\frac{1}{2} & 1 & 0 & . & . \\ 0 & \binom{5}{2}\frac{1}{6} & 0 & \binom{5}{4}\frac{1}{2} & 1 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . \\ 0 & \frac{3}{2} & 1 & 0 & 0 & . & . \\ \frac{4}{6} & 0 & 2 & 1 & 0 & . & . \\ 0 & \frac{10}{6} & 0 & \frac{5}{2} & 1 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

Recall the definition of generalized Bernoulli polynomials introduced in section 7. In [22], Zhizhang and Wang define an analogous *generalized Bernoulli matrix*, $\mathscr{B}^{(\alpha)}(x)$, defined at the $(i, j) - \text{th}$ entry by:

$$B^{(\alpha)}_{i,j}(x) = \begin{cases} \binom{i}{j} B^{(\alpha)}_{i-j}(x) & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases}$$

These matrices have a variety of useful properties and relationships with other named matrices, including Pascal's matrix, the Fibonacci matrix and Vandermonde's matrix [22].

14 The Regular Primes

We end this primer with an application of the Bernoulli numbers to one of the great solved problems of mathematics: the simply-stated Fermat's Last Theorem.

Theorem 14.1. *The equation*

$$x^n + y^n = z^n$$

has no integer solutions x, y, z *for positive integers* $n > 2$.

The theorem was proven in 1994 by Andrew Wiles (and hundreds of other mathematicians who contributed important results). But before his definitive proof, one of the most promising results–and one which would form the basis for the eventual solution–was developed by the German mathematician Ernst Kummer over a century prior.

Kummer's result was the product of another mathematician's mistake. Kummer had spent little time on Fermat's Last Theorem, which he considered a "curiosity of number theory rather than a major item," until March 1847, when the French mathematician Gabriel Lamé published a "complete proof" of the theorem.

Lamé's main contribution was noticing the sum $x^n + y^n$ could be decomposed into factors involving the n roots of unity:

$$(x+y)^n = (x+y)(x+\zeta y)(x+\zeta^2 y)\cdots(x+\zeta^{n-2}y)(x+\zeta^{n-1}y)$$

This was a useful step; however, he incorrectly assumed that this factorization was unique in $\mathbb{Q}(\zeta_p)$. But Kummer himself had proven years prior that this was not the case. Kummer felt compelled to respond, and in the few weeks after Lamé's publication, he had a proof for a select group of integers n that would satisfy Fermat's Last Theorem. He called them the "regular primes."

Definition 14.1. Odd prime p is a *regular prime* if the class number of $\mathbb{Q}(\zeta_p)$ is relatively prime to p.

For the reader who is unfamiliar with class numbers, this definition may be confusing. By definition, a *class number* is the order of the ideal class group $\mathbb{Z}[\zeta_p]$. But more intuitively, the class number can be understood as "a scalar quantity describing how 'close' elements of a ring of integers are to having unique factorization" [17]. If the class number is 1, then the ring has unique factorization. For positive values greater than 1, the closer to 1 the class number is the 'closer' to having prime factorization.

Kummer proved an equivalent definition, which almost by magic, involves the Bernoulli numbers:

Definition 14.2. A *regular prime* p is an integer such that it does not divide the numerator of $B_2, B_4, B_6, \ldots B_{p-3}$.

Kummer proved Fermat's Last Theorem for all regular primes. This, of course begs the question: how many regular primes are there? We know that the first irregular prime is 37, because

$$B_{32} = -\frac{7709321041217}{510} = 37 \cdot \frac{208360028141}{510}.$$

Beyond that, we know that there are infinitely many irregular primes, but it is not known if there are infinitely many *regular* primes [13]. Computational studies have shown that about 60% of primes are regular, and mathematician Carl Siegel has conjectured that the exact proportion converges to $e^{-1/2}$. However, neither hypothesis has been confirmed.

Regardless, Kummer's early work into Fermat's Last Theorem paved the way for mathematicians of the twentieth century to finish off the problem.

15 Conclusion

The Bernoulli numbers are a powerful and wonderfully surprising sequence. While they were originally discovered in the process of summing the integer powers, mathematicians have uncovered the Bernoulli numbers in fields across mathematics. In this primer, we were able to discuss the history of the Bernoulli numbers, including the work of mathematicians across continents and centuries. We considered the independent discoveries of the sequence by Seki Takakazu and Jakob Bernoulli and emphasized that both deserve recognition for their achievements. We proceeded to follow in Jakob Bernoulli's steps to find a formula for the sums of integer powers. Then we defined the sequence, made some preliminary observations, and delved into the applications of the Bernoulli numbers to important problems in mathematics. We ended with several direct formulas, as well as appearances of the Bernoulli numbers in matrix algebra and Fermat's Last Theorem.

Despite the breadth of this primer, there is much more to discuss, including an intriguing identity by the self-taught genius Srinivasa Ramanjuan [15], fascinating investigations into the numerators and denominators of the Bernoulli numbers by Georgii Voronoi [13], "curious" and "exotic" identities of the sequence [21], and useful generalizations of the Bernoulli numbers to q-Bernoulli numbers by Leonard Carlitz [4]. The Bernoulli numbers also lead easily into discussions of the Euler numbers, computational methods, and analytic continuation.

There is simply too much to cover. In fact, the literature on the Bernoulli numbers is so voluminous that several bibliographies have been developed over the past decades to help mathematicians navigate research studies related to the sequence. One bibliography developer, Professor Karl Dilcher, who helped to write *"Bernoulli numbers. Bibliography (1713-1990),"* continued to collect and maintain references to all Bernoulli-related publications well into the 2000s. As of 2005, the bibliography contained 2970 entries by 1493 authors [9]. We encourage the curious reader to explore these papers, or the other papers, websites, and textbooks cited in this primer. The sheer scale of research on this sequence demonstrates what we hope the reader has begun to feel intuitively about the Bernoulli numbers: that you can define them, but never fully understand them.

We conclude this primer with an anecdote. The Whitman mathematics department is concentrated in the second-floor hallway of Olin Hall where most mathematics professors have their offices. Hanging from the wall in the hallway is a beautiful geometric print that I pass by nearly every day and often stop to look at. The print is captioned with a quote by the British philosopher, mathematician, writer, and Nobel laureate Bertrand Russell, a true practitioner of liberal arts and critical thought: "Mathematics, rightly viewed, possesses not only truth but also supreme beauty."

It's hard to say it better than that.

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17 Appendix: Notation and Definitions

The Bernoulli numbers were discovered over 300 years ago. It makes sense that notation of the numbers has diverged somewhat since the initial discovery. For example, we have already noted that there are two definitions for the Bernoulli polynomial that mathematicians use.

In this section, we explore some of the differences between the way in which the sums of powers, the Bernoulli numbers, and the Euler-Maclaurin Summation Formula are expressed. We hope that this will help the adventurous undergraduate reader to navigate several of the differences in notation that they may encounter.

17.1 The Sums of Powers

The function $S_m(n)$ is often used to denote the sums of integer powers. There are two common ways to define the function. We define it to be:

$$S_m(n) = 1^m + 2^m + 3^m + \cdots + (n-1)^m + n^m$$

which we believe to be fairly intuitive. However, some others say

$$S_m(n) = 1^m + 2^m + 3^m + \cdots + (n-1)^m$$

because more often than not, we consider $S_m(n-1)$ rather than $S_m(n)$ in calculations using the Bernoulli numbers (such as in 2). We hope the reader stays aware of these two conventions.

17.2 The First and Only Odd Bernoulli Number: B_1

The first odd Bernoulli number B_1 is a unique case as it is the only non-trivial odd number. S.C. Woon, in his paper "Analytic Continuation of Bernoulli Numbers, a New Formula for the Riemann Zeta Function, and the Phenomenon of Scattering of Zeros" outlines a reason why we may want to reconsider the definition of the Bernoulli numbers [20].

In the paper, Woon develops what is called an *analytic continuation* of the Bernoulli numbers, a technique to extend the domain of an analytic function. Without delving too far into the details, Woon uses this method to derive a unique curve that passes through all of the Bernoulli numbers, connecting each of them and *continuing* into the negative real numbers. The curve could allow mathematicians to consider Bernoulli numbers with rational, irrational, and negative indices.

There is one issue with the continuation. In the continuation, $B_1 = \frac{1}{2}$ rather than $-\frac{1}{2}$. However, Woon notes that Euler's original sign convention for the Bernoulli numbers "was actually arbitrary" [20], and so perhaps we need to reconsider the original definition of the sequence.

Woon suggests that the Bernoulli numbers should be redefined as follows:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} (-1)^n \frac{B_n x^n}{n!}$$

for $|x| < 2\pi$. This redefinition only actually affects the one nontrivial odd Bernoulli number, B_1, which would flip sign to $\frac{1}{2}$. As he states:

> Here's my little appeal to the Mathematics, Physics, Engineering, and Computing communities to introduce the missing signs into the sum in the definition of Bernoulli numbers ... because the analytic continuation of Bernoulli numbers fixes the arbitrariness of the sign convention of B1.

Will this convention change in the future? Who knows? But this is a good thing to consider when looking through the literature.

17.3 The Euler-Maclaurin Summation Formula

Note that there are several ways to write the famous Euler-Maclaurin Summation Formula. In different papers, one could easily come to believe that these expressions are different, but they are all equivalent. Here are some examples of the forms you may see:

1. Our definition was:
$$\sum_{i=a}^{b-1} f(i) = \int_a^b f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_a^b + R_m$$
where R_m, the remainder, is equal to $(-1)^{m+1}\int_a^b \frac{B_m(y-\lfloor y \rfloor)}{m!} f^{(m)}(x)dx$ and tends towards zero as m approaches infinity.

2. Another elementary way to write the formula is [14]:
$$\sum_{n=a}^{b} f(n) = \int_a^b f(t) + \frac{1}{2}(f(b)+f(a)) + \sum_{i=2}^{k} \frac{b_i}{i!}(f^{(i-1)}(b) - f^{(i-1)}(a)) - \int_a^b \frac{B_k(\lfloor 1-t \rfloor)}{k!} f^{(k)}(t).$$

3. Some mathematicians define the *periodic Bernoulli function* $\bar{B}_n(x) = B(\lfloor x \rfloor)$ and let $P_k(x) = \frac{\bar{B}_k(x)}{k!}$, giving an expression that looks slightly different [14]:
$$\sum_{n=a}^{b} f(n) = \int_a^b f(t) + \frac{1}{2}(f(b)+f(a)) + \sum_{i=2}^{k} \frac{b_i}{i!}(f^{(i-1)}(b) - f^{(i-1)}(a)) - \int_a^b P_k(x) f^{(k)}(t).$$

4. You may notice that the above formulas consider B_1 separately. Then the right hand summation is taken over the even Bernoulli numbers and the EMSF becomes [15]:
$$\sum_{k=a}^{b} f(k) \int_a^b f(x) + \frac{1}{2}(f(a)+f(b)) + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) + R_n.$$

5. Furthermore, in a proof given in [12], we can see that
$$R_n \leq \frac{2}{(2\pi)^{2p}} \int_a^b |f^{(2p+1)}(x)|.$$

These alternative ways of expressing the Euler-Maclaurin Summation Formula may be confusing, but each is valid. The fact that so many forms exist is unsurprising. Each mathematician may perfer their own form of any number of formulas, so it is up to the mathematical reader to make the connections between each version.

17.4 The Bernoulli Matrix

The Bernoulli matrix defined in this primer should not be confused with the *random Bernoulli matrix*, often shortened to "Bernoulli matrix." The latter is a matrix such that every entry is -1 or 1, with an equal probability for either. These matrices are more common than the Bernoulli matrices we define, so the reader should be aware in case they explore further.

18 Appendix: List of Bernoulli Numbers

We have found that this sequence is surprising and applicable to such a variety of situations, it is easy to lose the fact that we are talking about a very real sequence of rational numbers.

Feel free to refer to this list of Bernoulli numbers in order to do test calculations or convince yourself of certain properties. For brevity, we exclude all trivial odd Bernoulli numbers.

$$B_0 = 1$$
$$B_1 = -1/2$$
$$B_2 = 1/6$$
$$B_4 = -1/30$$
$$B_6 = 1/42$$
$$B_8 = -1/30$$
$$B_{10} = 5/66$$
$$B_{12} = -691/2730$$
$$B_{14} = 7/6$$
$$B_{16} = -3617/510$$
$$B_{18} = 43867/798$$
$$B_{20} = -174611/330$$
$$B_{22} = 854513/138$$
$$B_{24} = -236364091/2730$$
$$B_{26} = 8553103/6$$
$$B_{28} = -23749461029/870$$
$$B_{30} = 8615841276005/14322$$
$$B_{32} = -7709321041217/510$$
$$B_{34} = 2577687858367/6$$
$$B_{36} = -26315271553053477373/1919190$$
$$B_{38} = 2929993913841559/6$$
$$B_{40} = -261082718496449122051/13530$$
$$B_{42} = 1520097643918070802691/1806$$
$$B_{44} = -27833269579301024235023/690$$
$$B_{46} = 596451111593912163277961/282$$
$$B_{48} = -5609403368997817686249127547/46410$$
$$B_{50} = 495057205241079648212477525/66$$
$$B_{52} = -801165718135489957347924991853/1590$$
$$B_{54} = 29149963634884862421418123812691/798$$
$$B_{56} = -2479392929313226753685415739663229/870$$
$$B_{58} = 84483613348880041862046775994036021/354$$
$$B_{60} = -1215233140483755572040304994079820246041491/56786730$$
$$B_{62} = 12300585434086858541953039857403386151/6$$
$$B_{64} = -106783830147866529886385444979142647942017/510$$
$$B_{66} = 1472600022126335654051619428551932342241899101/64722$$
$$B_{68} = -78773130858718728141909149208474606244347001/30$$