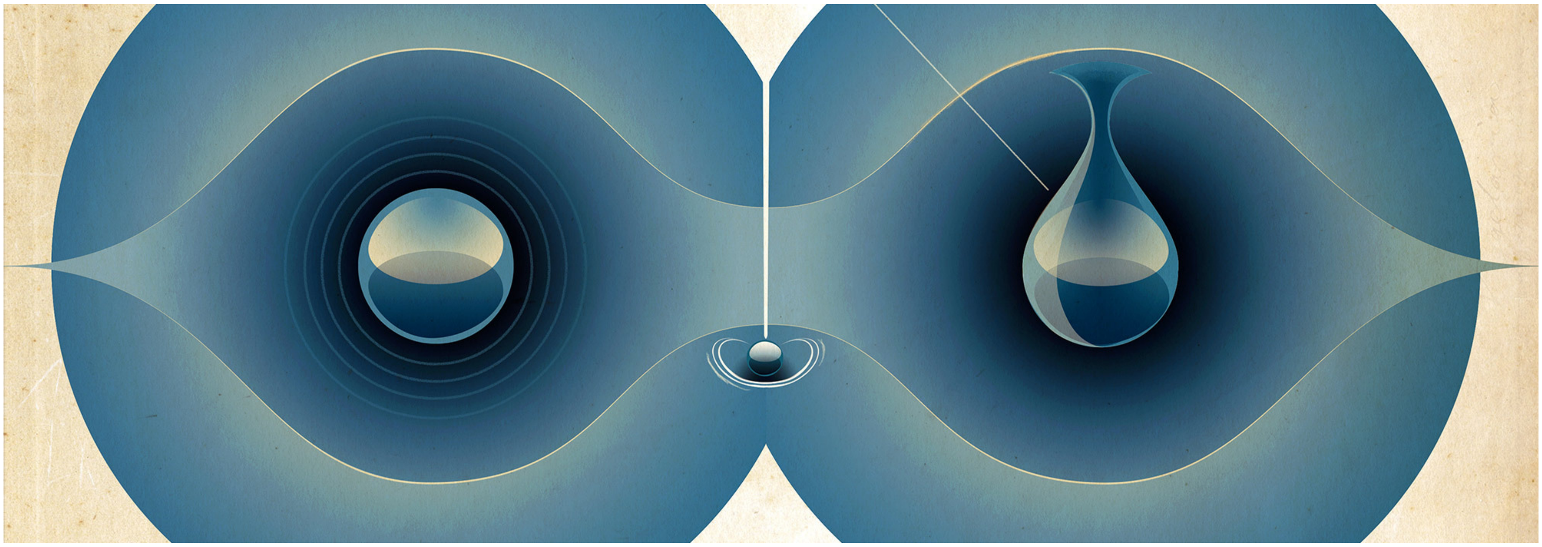

Imaginary numbers are real

These odd values were long dismissed as bookkeeping. Now physicists are proving that they describe the hidden shape of nature

by Karmela Padavic-Callaghan

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Many science students may imagine a ball rolling down a hill or a car skidding because of friction as prototypical examples of the systems physicists care about. But much of modern physics consists of searching for objects and phenomena that are virtually invisible: the tiny electrons of quantum physics and the particles hidden within strange metals of materials science along with their highly energetic counterparts that only exist briefly within giant particle colliders.

In their quest to grasp these hidden building blocks of reality scientists have looked to mathematical theories and formalism. Ideally, an unexpected experimental observation leads a physicist to a new mathematical theory, and then mathematical work on said theory leads them to new experiments and new observations. Some part of this process inevitably happens in the physicist's mind, where symbols and numbers help make invisible theoretical ideas visible in the tangible, measurable physical world.

Sometimes, however, as in the case of imaginary numbers – that is, numbers with negative square values – mathematics manages to stay ahead of experiments for a long time. Though imaginary numbers have been integral to quantum theory since its very beginnings in the 1920s, scientists have only recently been able to find their physical signatures in experiments and empirically prove their necessity.

In December of 2021 and January of 2022, two teams of physicists, one an international collaboration including researchers from the Institute for Quantum Optics and Quantum Information in Vienna and the Southern University of Science and Technology in China, and the other led by scientists at the University of Science and Technology of China (USTC), showed that a version of quantum mechanics devoid of imaginary numbers leads to a faulty description of nature. A month earlier, researchers at the University of California, Santa Barbara reconstructed a quantum wave function, another quantity that cannot be fully described by real numbers, from experimental data. In either case, physicists cajoled the very real world they study to reveal properties once so invisible as to be dubbed imaginary.

For most people the idea of a number has an association with counting. The number five may remind someone of fingers on their hand, which children often use as a counting aid, while 12 may make you think of buying eggs. For decades, scientists have held that some animals use numbers as well, exactly because many species, such as chimpanzees or dolphins, perform well in experiments that require them to count.

Counting has its limits: it only allows us to formulate so-called natural numbers. But, since ancient times, mathematicians have known that other types of numbers also exist. Rational numbers, for instance, are equivalent to fractions, familiar to us from cutting cakes at birthday parties or divvying up the cheque after dinner at a fancy restaurant. Irrational numbers are equivalent to decimal numbers with no periodically repeating digits. They are often obtained by taking the square root of some natural numbers. While writing down infinitely many digits of a decimal number or taking a square root of a natural number, such as five, seems less real than cutting a pizza pie into eighths or 12ths, some irrational numbers, such as pi, can still be matched to a concrete visual. Pi is equal to the ratio of a circle's circumference and the diameter of the same circle. In other words, if you counted how many steps it takes you to walk in a circle and come back to where you started, then divided that by the number of steps you'd have to take to make it from one point on the circle to the opposite point in a straight line passing through the centre, you'd come up with the value of pi. This example may seem contrived, but measuring lengths or volumes of common objects also typically produces irrational numbers; nature rarely serves us up with perfect integers or exact fractions. Consequently, rational and irrational numbers are collectively referred to as 'real numbers'.

Negative numbers can also seem tricky: for instance, there is no such thing as 'negative three eggs'. At the same time, if we think of them as capturing the opposite or inverse of some quantity, the physical world once again offers up examples. Negative and positive electric charges correspond to unambiguous, measurable behaviour. In the centigrade scale, we can see the difference between negative and positive temperature since the latter corresponds to ice rather than liquid water. Across the board then, with positive and negative real numbers, we are able to claim that numbers are symbols that simply help us keep track of well-defined, visible physical properties of nature. For hundreds of years, it was essentially impossible to make the same claim about imaginary numbers.

In their simplest mathematical formulation, imaginary numbers are square roots of negative numbers. This definition immediately leads to questioning their physical relevance: if it takes us an extra step to work out what negative numbers mean in the real world, how could we possibly visualise something that stays negative when multiplied by itself? Consider, for example, the number +4. It can be obtained by squaring either 2 or its negative counterpart -2. How could -4 ever be a square when 2 and -2 were both already determined to produce 4 when squared? Imaginary numbers offer a resolution by introducing the so-called imaginary unit *i*, which is the square root of -1. Now, -4 is the square of 2*i* or -2*i*, emulating the properties of +4. In this way, imaginary numbers are like a mirror image of real numbers: attaching *i* to any real number allows it to produce a square exactly the opposite of the one it was generating before.

Western mathematicians started grappling with imaginary numbers in earnest in the 1520s when Scipione del Ferro, a professor at the University of Bologna in Italy, set out to solve the so-called cubic equation. One version of the challenge, later referred to as the irreducible case, required taking the square root of a negative number. Going further, in his book *Ars Magna* (1545), meant to summarise all of algebraic knowledge of the time, the Italian astronomer Girolamo Cardano declared this variety of the cubic equation to be impossible to solve.

Almost 30 years later, another Italian scholar, Rafael Bombelli, introduced the imaginary unit *i* more formally. He referred to it as *più di meno*, or 'more of the less', a paradoxical phrase in itself. Calling these numbers imaginary came later, in the 1600s, when the philosopher René Descartes argued that, in geometry, any structure corresponding to imaginary numbers must be impossible to visualise or draw. By the 1800s, thinkers such as Carl Friedrich Gauss and Leonhard Euler included imaginary numbers in their studies. They discussed complex numbers made up of a real number added to an imaginary number, such as 3+4*i*, and found that complex-valued mathematical functions have different properties than those that only produce real numbers.

Yet, they still had misgivings about the philosophical implications of such functions existing at all. The French mathematician Augustin-Louis Cauchy wrote that he was 'abandoning' the imaginary unit 'without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it.'

In physics, however, the oddness of imaginary numbers was disregarded in favour of their usefulness. For instance, imaginary numbers can be used to describe opposition to changes in current within an electrical circuit. They are also used to model some oscillations, such as those found in grandfather clocks, where pendulums swing back and forth despite friction. Imaginary numbers are necessary in many equations pertaining to waves, be they vibrations of a plucked guitar string or undulations of water along a coast. And these numbers hide within mathematical functions of sine and cosine, familiar to many high-school trigonometry students.

At the same time, in all these cases imaginary numbers are used as more of a bookkeeping device than a stand-in for some fundamental part of physical reality. Measurement devices such as clocks or scales have never been known to display imaginary values. Physicists typically separate equations that contain imaginary numbers from those that do not. Then, they draw some set of conclusions from each, treating the infamous *i* as no more than an index or an extra label that helps organise this deductive process. Unless the physicist in question is confronted with the tiny and cold world of quantum mechanics.

Quantum theory predicts the physical behaviour of objects that are either very small, such as electrons that make up electric currents in every wire in your home, or millions of times colder than the insides of your fridge. And it is chock-full of complex and imaginary numbers.

> Imaginary numbers went from a problem seeking a solution to a solution that had just been matched with its problem

Emerging in the 1920s, only about a decade after Albert Einstein's paradigm-shifting work on general relativity and the nature of spacetime, quantum mechanics complicated almost everything that physicists thought they knew about using mathematics to describe physical reality. One big upset was the proposition that quantum states, the fundamental way in which objects that behave according to the laws of quantum mechanics are described, are by default complex. In other words, the most generic, most basic description of anything quantum includes imaginary numbers.

In stark contrast to theories concerning electricity and oscillations, in quantum mechanics a physicist cannot look at an equation that involves imaginary numbers, extract a useful punchline, then forget all about them. When you set out to try and capture a quantum state in the language of mathematics, these seemingly impossible square roots of negative numbers are an integral part of your vocabulary. Eliminating imaginary numbers would highly limit how accurate of a statement you could make.

The discovery and development of quantum mechanics upgraded imaginary numbers from a problem seeking a solution to a solution that had just been matched with its problem. As the physicist and Nobel laureate Roger Penrose noted in the documentary series *Why Are We Here?* (2017): '[Imaginary numbers] were there all the time. They've been there since the beginning of time. These numbers are embedded in the way the world works at the smallest and, if you like, most basic level.'

The complex object at the heart of all of quantum mechanics is the so-called wave function. It reflects a striking fundamental truth uncovered by quantum researchers – that everything, no matter how solid or corpuscular it seems, sometimes behaves like a wave. And it works the other way as well: electrons, the stuff of waves, can behave like particles.

'Louis de Broglie speculated that maybe these seemingly disparate features, undulatory and corpuscular, form a union not only in light but in everything,' writes Smitha Vishveshwara, a physicist at the University of Illinois Urbana-Champaign in her forthcoming book, 'Two Revolutions: Einstein's Relativity and Quantum Physics'. 'Maybe the stuff we're made of, which we know to be composed of particles, can have wavy traits,' she adds, paraphrasing the question that led the founders of quantum theory to make the complex-valued wave function the fundamental building block of their model of nature.

To determine the exact details of a quantum-mechanical wave function that describes some physical object, for example an electron moving within a metal, researchers turn to the Schrödinger equation. Named after the Austrian physicist Erwin Schrödinger, another architect of quantum theory's foundations, this equation accounts not only for the kind of tiny particle one is trying to describe, but also its environment. Is the electron seeking a less energetic and more stable state like a ball rolling down a steep hill? Has it received an energy 'kick' and is consequently executing a fast and complex motion like a football thrown in a spiral by a very strong athlete? The mathematical form of the Schrödinger equation allows for this information to be taken into account. In this way, the Schrödinger equation is directly informed by the particle's immediate physical reality. Nevertheless, its solution is always the wave function that inextricably contains imaginary numbers. Even Schrödinger was disturbed by this. In 1926, he wrote to his colleague Hendrik Lorentz, saying that: 'What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers.'

Today, almost a century after Schrödinger first voiced his concern, three independent teams of physicists have cornered imaginary numbers in their labs.

In the first experiment, researchers from the University of California, Santa Barbara (UCSB) and Princeton University went after the quantum wave function itself. Their work, appearing in the journal *Natur*e, demonstrated a first-of-its-kind reconstruction of the quantum-mechanical wave function from a laboratory measurement. The researchers experimentally studied how the semiconductor material gallium arsenide behaves after being exposed to a very fast pulse of laser light. More specifically, gallium arsenide re-emits some of the light that a laser shines onto it, and the UCSB team was able to show that, remarkably, properties of that light depend not only on the details of the wave functions of particles inside the material, but in particular on the imaginary parts of those wave functions.

Semiconductors such as gallium arsenide take up the middle ground between conducting materials, where electrons form rivers of moving charges that we call currents, and insulators, which hold on to their electrons so tightly that the formation of a current is impossible. In a semiconductor, most electrons do stay put, but here and there a few can start moving around, constituting tiny currents. An odd feature of this type of conduction is that every electron that manages to move gains a partner automatically – a particle-like entity called a 'hole', which carries positive electric charge. If the electron were a droplet of water in a pond, the existence and motion of the hole would be like the vacancy left after the droplet is removed, gaining a life of its own. Both electrons and their partner holes follow the rules of quantum mechanics, so the best way that physicists have of describing them is to write down a wave function for each.

An important part of every such wave function is its phase, which contains an imaginary number. Often, it reflects interactions that a quantum particle may have experienced while travelling along some path in space. Two wave functions can overlap and combine just like two waves on the surface of water, and the resulting ripple pattern, which in the quantum case informs scientists of where particles corresponding to those wave functions are most likely to be, depends on the wave functions' phases. In the UCSB and Princeton experiment, the phases of the wave functions of gallium arsenide's holes and electrons also dictated what kind of light the material could re-emit.

To uncover that connection, researchers first gave electrons in the material an energy boost by shining a fast pulse of near-infrared laser light. This energy boost made the electrons move through the material and created their companion holes. The physicists used another laser to briefly separate the two kinds of particles. After a short time of lonely motion through the semiconductor, the electron and hole pairs were allowed to reunite. Because both particles acquired energy while they were moving alone, their reunion resulted in a flash of light. Researchers determined the imaginary wave-function phase for the holes involved in this process by measuring that light – which was a concrete entity in the natural world.

Other physicists, meanwhile, now wonder whether theories can be reconfigured to avoid the apparent conflict between the real and the imaginary. In this view, instead of looking for imaginary numbers in the lab, physicists just need to find a different labelling system, one that requires real numbers only. This type of theory is known as 'real quantum mechanics'.

> Some conclusions can never be reached without imaginary numbers

Historically, real quantum mechanics has had not only proponents but also some successes in the realm of mathematical proofs and investigations. Theorists have been able to show that certain properties of quantum-mechanical systems can indeed be captured without resorting to imaginarity. Within the last year, however, a new crop of proofs and experiments proved that this line of reasoning can only go so far. Laboratory experiments involving quantum computers and quantised light now strongly indicate that imaginary and complex numbers are an indispensable part of the quantum, and therefore our own, world.

The theoretical work, spearheaded by physicists at the Austrian Academy of Sciences in Vienna, and the experiments that put it to the test in laboratories in both Austria and China, approach the issue through a kind of game.

In the theoretical study, the 'players' are three imaginary physicists, Alice, Bob and Charlie, who use quantum states as their board-game pieces and a series of sophisticated quantum operations as their in-game moves. At the end of the game, the three can compare notes on what properties their quantum state acquired during play. The Vienna physicists showed that some conclusions can never be reached without imaginary numbers. It was as if they had found that real quantum theory could not help a sports analyst predict that a basketball player successfully shooting the basket from the three-point arc would score their team the full three points.

Such game-like tests of competing theories of nature are something of a tradition in quantum mechanics. They date back to the Northern Irish physicist John Bell who, in the 1960s, used a similar approach to prove that quantum mechanics itself is truly necessary for an accurate description of nature. In this case, physicists pitted quantum mechanics against classical physics, which dates all the way back to Isaac Newton, and found that the former always excelled in predicting the outcomes of their experiments.

This approach, dubbed the Bell test, included only two 'players', Alice and Bob, who could not make sense of their post-game results unless they viewed them through the lens of quantum theory. Classical physics, researchers concluded, simply was not the best description of the world. Miguel Navascués, a physicist at the Austrian Academy of Sciences and co-author of both experimental and theoretical studies of the new Bell game noted that his team's effort provided a way to make exactly the same evaluation of real and complex-value quantum theories. 'If you can conduct this experiment,' he said, 'then you will have refuted real-number quantum physics.'

In the experiment carried out at USTC, the Bell game took place inside a quantum computer, where superconducting units called 'qubits' were controlled by microwave pulses. In the experiment that Navascués was involved with, the arena was an optical setup where researchers worked with quantum light – in other words, a stream of photons that could be altered by beam-splitters and other lab equipment.

In either case, the outcome of the game was impossible to predict accurately by any version of quantum physics that renounced complex numbers. Not only did physicists infer that imaginary numbers can indeed show up in experiments, but that, even more strikingly, they had to be considered in order for experiments in the quantum realm to be understood correctly at all.

The studies mentioned here carry important implications for the most heady and profound ideas about quantum mechanics and the nature of physical reality. They are also important milestones for the development of new quantum technologies. Manipulating wave functions and wave-function phases is an important tool in quantum information and quantum computing. Accordingly, the UCSB experiment may help advance device design in those fields. 'If you're thinking about building any sort of device that takes advantage of quantum mechanics, you're going to need to know its [wave function's] parameters really well,' Joe Costello, a physics PhD student at UCSB and the lead author on the study, emphasised when discussing the work.

Similarly, when scientists write algorithms that deal with quantum information, they must consider whether there are any advantages to using complex-valued quantum states. Recent works led by USTC and Vienna strongly suggest the answer is 'yes'. Quantum computers will ultimately vastly surpass their conventional counterparts, making the development of best algorithmic practices a critical task. Almost a hundred years after Schrödinger bemoaned imaginary numbers, physicists are finding they may be useful in very practical ways.

> Quantum physics has revealed that we've misunderstood imaginary numbers all along

In his book *The Road to Reality* (2004), Penrose writes that: 'In the development of mathematical ideas, one important initial driving force has always been to find mathematical structures that accurately mirror the behaviour of the physical world.' In this way, he summarises the trajectory of theoretical physics overall. Notably, he adds that ‘in many instances, this drive for mathematical consistency and elegance takes us to mathematical structures and concepts which turn out to mirror the physical world in a much deeper and more broad-ranging way than those that we started with.’ Imaginary numbers have transcended their original place as mere placeholders, transforming our grasp of reality and illuminating this grand idea.

Quantum theory has historically challenged many seemingly ‘common sense’ assumptions about nature. It has, for example, changed the way physicists think about an experimenter’s ability to measure something with certainty, or the claim that objects can be affected only by other objects in their immediate surroundings. When quantum theory was first formulated, it scandalised many luminaries of science at the time, including Einstein who contributed to its foundations himself. Working with quantum ideas and poking quantum systems has always, by default, come with the possibility of uncovering something unexpected at best, and bizarre at worst. Now quantum physics has revealed that we’ve misunderstood imaginary numbers all along. They may have, for a time, seemed to be just a mental device inhabiting the minds of physicists and mathematicians, but since the real world that we inhabit is indeed quantum, it’s no surprise that imaginary numbers can be found, quite clearly, within it.